Applying stochastic volatility models for pricing and hedging derivatives

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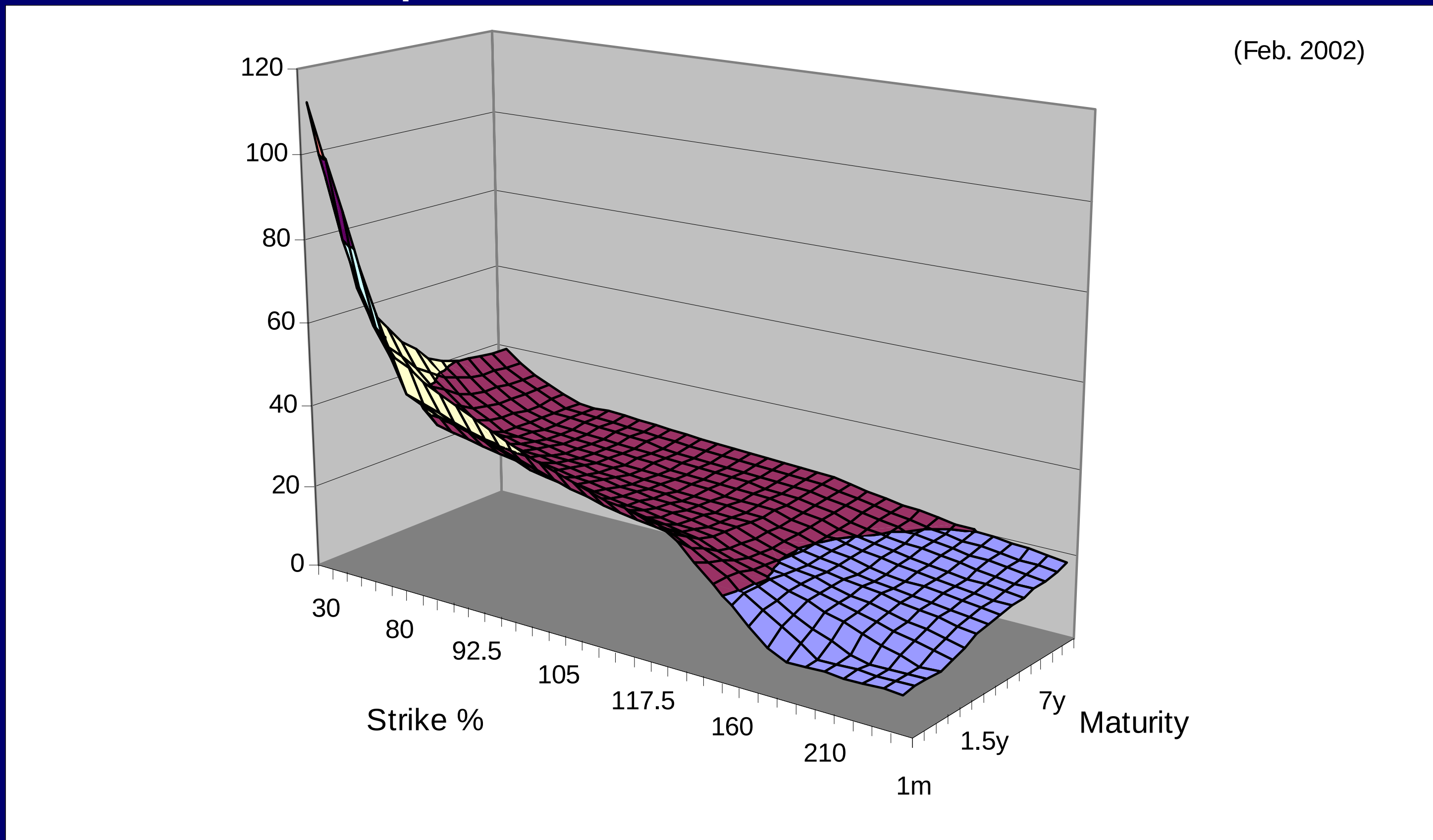
Outline

- Using stochastic volatility models for pricing
  - understanding the concepts and issues underlying the theory
- Reviewing stochastic volatility models
  - Heston, Hull-White, Local Volatility and others
- Implementing stochastic volatility models
  - issues of completeness
  - calibrating the models
- Pricing and hedging with stochastic volatility models
- Assessing the impact of stochastic volatility on pricing and hedging exotic options
- Practical worked example: Cliquets

Empirical evidence for the smile

STOXX50 implied volatilities



Characteristics of equity smile

- Features of implied volatility surface:
  - Implied vol decreases with increasing strike.
  - Vol skew decays with time to maturity.
  - The shape depends on the market.

- Why does the smile exists ?
  - There is asymmetry in risk assessment:
    A big down move is more “risky” than a big up move for a stock holder.
  - Correlation between volatiltiy and stock is negative.

Stochastic volatility and exotic options

- Many exotic payoffs are sensitive to volatility skew (examples: Barrier options, Flexi-baskets, Altiplano).
- Using a stochastic volatility model returns prices that take this risk into account.
- Clearly, this must also be taken into consideration in the derivation of a hedging strategy.

Stochastic volatility models

- The concepts and issues underlying the theory
- Pricing in stochastic volatility models
- Hedging in stochastic volatility models

Concepts and issues

- "Stochastic volatility" (SV) denotes a class of models where the stock price is modelled as

$$dS_t = r_t S_t dt + \sigma_t S_t dW_t$$

where σ is by itself a stochastic process and satisfies a SDE

$$d\sigma_t = \lambda_t dt + \zeta_t dW_t^\sigma$$

- W and W^σ have correlation ρ

- The integrands λ and ξ are typically determined by a set $\underline{\sigma}$ of model parameters.
These parameters are obtained via calibration.

Concepts and issues

- Issues to address:
  - price European options efficiently and accurately to be able to calibrate
  - price exotic structures and options
  - derive hedging strategies consistent with the model (market incompleteness)
  - integrate in global risk framework

Pricing in stochastic volatility models

- European options are efficiently and accurately priced by Fourier inversion (FFT), see (Carr and Madan, 1999)
- The method is efficient and allows calibration.
- It can also be used to price to forward started options.

Calibration

- Fast pricing of Europeans allows calibration to the implied volatility surface ie the estimation of the model parameters $\underline{\sigma}$.
- This means minimizing an objective function, e.g.

$$F(\underline{\sigma}) = \sum_{i=1}^{N} w_i \left(H_i(\underline{\sigma}) - P_i\right)^2, \underline{\sigma} = \text{model parameters, } P_i = \text{market prices}$$

- In fact, the above minimization is too naïve and leads to contradictions ($\rightarrow$ worked example later).
  One should use a penalized objective function to avoid this

$$F_p(\underline{\sigma}) = F(\underline{\sigma}) + p(\underline{\sigma})$$

Pricing exotic options

- The dynamics is assumed under the valuation probability P. Calibration allows to discover the values of the model parameters $\underline{\sigma}$.

- Once the model is calibrated, the arbitrage price at time t of an exotic option with payoff O is given by the usual

$$\mathrm{E}\left[e^{-\int r_s \mathrm{d}s} O \mid \mathrm{F}_t \right]$$

Pricing exotic options

- Numerical methods can be implemented
  - Monte Carlo simulation
  - 2-factor finite differences
  - 2-factor trees

- Multi-underlying options can also be valued, by assuming each of them to follow an SV model. This typically requires extensive Monte Carlo.

Hedging complex options

- Stochastic volatility models are incomplete in the sense that not all optional payoffs can be replicated by trading solely in the stock.
  - Mathematically, this stems from the use of 2 Brownians.
  - Economically, this is inherent to taking into account option prices information!
- A hedging strategy must then use all liquid assets, including vanilla options.
  - In theory, it is enough to trade in one additional non-pathological option with a longer maturity than the structure.

Hedging complex options

- Select European options with characteristics close to the the option to hedge. In general, there should be as many such Europeans as parameters for the stochastic volatility in order to make the market complete.
- A delta-hedge strategy with respect to the parameters of the stochastic volatility model these options and the stock can then be computed
- The selection can be done by an optimisation procedure, with respect to quality criteria of a hedging strategy. For example, one can minimise the corresponding Gammas.

Integration in VaR

- A Black-Scholes VaR engine requires Black-Scholes sensitivities to market observables.
- An SV model yields only sensitivities to its parameters.
- We need to convert between these two representations.

- Assume we have optimised N calibration instruments in the model parameters $\underline{\sigma}=(\sigma_1,\ldots,\sigma_M)$. We have minimized:

$$\sum_{i=1}^{N} w_i \left(H_i(\underline{\sigma}) - P_i\right)^2 + p(\underline{\sigma})$$

- ie we have for all j:

$$\sum_{i=1}^{N} 2w_i \left(H_i(\underline{\sigma}) - P_i\right)\frac{\partial H_i}{\partial \sigma_j} + \frac{\partial p}{\partial \sigma_j} = 0$$

Integration in VaR

- We will assume that the sensitivity of each calibration instrument to each parameter is constant near the optimum:

$$A_{ji} = 2w_i \frac{\partial H_i}{\partial \sigma_j} \equiv \text{constant}$$

- Differentiate w.r.t. P_k, since the penalty function does not depend on market prices

$$\sum_{i=1}^{N} A_{ji} \frac{\partial H_i}{\partial P_k}(\underline{\sigma}) = A_{jk}$$

- and apply the chain rule:

$$\frac{\partial H_i}{\partial P_k} = \sum_{\mu} \frac{\partial H_i}{\partial \sigma_\mu} \frac{\partial \sigma_\mu}{\partial P_k}$$

Integration in VaR

- This yields the matrix equation for the sensitivities of the model parameters to market prices:

$$\frac{\partial \sigma}{\partial P} = \left(A \frac{\partial H}{\partial \sigma} \right)^{-1} A$$

- Obtaining the sensitivities to B-S (implied) volatilities is just a matter of multiplying by B-S vegas:

$$\frac{\partial \underline{\sigma}}{\partial \sigma_k} = \frac{\partial \underline{\sigma}}{\partial P_k} \frac{\partial P_k}{\partial \sigma_k}$$

- Clearly, deltas need adjusting due to the sensitivity of model parameters to spot:

$$\frac{\partial}{\partial \underline{S}} V(S,\underline{\sigma}) = \frac{\partial V}{\partial \underline{x}}(S,\underline{\sigma}) + \frac{\partial V}{\partial \underline{\sigma}} \frac{\partial \underline{\sigma}}{\partial S}(S,\underline{\sigma})$$

Models review

- Hull & White (1987)
- Heston (1993)
- Local volatility, eg Derman - Kani (1994)
- Others

Models review: the Hull-White model (1987)

- The stock is modelled as:

$$dS_t = r_t S_t dt + \sigma_t S_t dW_t$$

where

$$d\sigma_t^2 = \kappa \sigma_t^2 dt + \zeta \sigma_t^2 dW_t^\sigma$$

is a log-normal process.

- Hull-White Parameters:
  - Vol return κ
  - Vol of vol ζ
  - Correlation ρ

One Artificial Parameter:

Short vol σ_0

Short vol is not observable and therefore turns into a parameter (which is part of the calibration).

Models review: the Heston model (1993)

- The stock is modelled as:

$$dS_t = r_t S_t dt + \sigma_t S_t dW_t$$

where

$$d\sigma_t^2 = \kappa\left(\theta - \sigma_t^2\right)dt + \zeta\sigma_t dW_t^\sigma$$

is a square-root process.

- Heston Parameters:
  - Reversion speed κ
  - Long vol $\sqrt{\theta}$
  - Vol of vol ζ
  - Correlation ρ

One Artificial Parameter:
Short vol σ_0

Models review: Local volatility

- A local volatility model has the form

$$dS_t = r_t S_t dt + \sigma(t, S_t) S_t dW_t$$

- ie volatility is a function $\sigma(t,x)$ of time and spot.
  - Functional types, eg CEV $\sigma(t, S_t) := \hat{\sigma}_t \cdot S_t^{\gamma_t}$
  - Implied types, eg Derman-Kani (1994)

- Local volatiliy models generally have instability problems.

Models review: other models

- Ornstein-Uhlenbeck process: Schobel and Zhu $d\sigma_t = \kappa(\theta - \sigma_t)dt + \zeta dW_t^\sigma$

- Levy-processes

- Jump models also give skewed implied volatilities

- Presentations on model comparisons can be found at *http://www.dbquant.com* ($\rightarrow$ “Conferences”).

The Heston model

- Pros
  - Mean-reversion
  - Simplicity
  - Hedging and integration in VaR
- Cons
  - Modeling a non-observable variable (v_0 has to be calibrated together with the parameters)
  - Implied skew is often too weak for short maturities
  - Unstable dynamics ($\rightarrow$ worked example)

Selecting the most appropriate model

- There is a compromise to do between
  - capturing the risks of the product to price and hedge
  - the computational effort required in using different methods
  - the institution's strategy regarding global risk management
- One should use a common model for managing a whole portfolio.
- European option prices are *static* information. Information on the dynamics is needed (e.g. prices of path-dependent options) to discriminate between models, but there is no consensus today as to what such information is available.

Practical example

- A simple structure: the cliquet.
  Pricing and hedging in the Heston model
- Study of implied forward volatilities
- The relevant variables and statistical properties
- Failure of the Heston models in some cases

Cliquet in the Heston model

- A cliquet is mainly a sum of forward started options.
- The simpliest cliquet structure has the payoff:

$$Cliquet = \sum_{i=1}^{N} \left(\frac{S_{t_{i+1}}}{S_{t_i}} - k \right)^{+}$$

- Global floors and caps are usually imposed.
- Often t_i-t_{i-1} is small (e.g. 3months) but the structure runs over long term (e.g. 5y).

Implied forward volatility

- Definitions
  - given a volatility term structure $\sigma(T)$, the *forward volatility* for the (future) period $[T_1,T_2]$ is given by

$$\sigma^2_{[T_1,T_2]} = \frac{1}{T_2 - T_1}\left(T_2\sigma^2_{T_2} - T_1\sigma^2_{T_1}\right)$$

  - the *implied forward volatility* is defined to be the number to be plugged into the Black-Scholes formula, to obtain the prices of forward-start options:

$$\left(\frac{S_{T_2}}{S_{T_1}} - k\right)^+$$

Forward implied volatility

- The danger in forward volatility:

  Example: T_1=1 year, σ_1=40%; T_2=2 years, σ_2=30%.
  This yields a forward volatility σ_{12}= 14.14%!

  With Heston, we calibrate to 6m, 1y, 2y and 4y and get a implied forward volatility of 28%.

Implied forward volatility

- Fourier inversion and the FFT can still be used (joint Fourier transform of $(\ln(S_t),\ln(S_T))$ to price forward-start options
- In the case of deterministic volatility, forward volatilities and implied forward volatilities coincide.
- For stochastic volatilities, this is not the case!

Implied forward volatility

- Forward volatilities vs. implied forward volatilies in the Heston model:

Start	Maturity	Forward volatility	Implied forward volatility	Absolute Difference (bps)
1m	1m	20.95	21.24	29.08
	3m	20.67	20.88	21.86
	6m	20.38	20.59	20.99
	1y	20.39	20.59	19.50
3m	1m	20.39	20.20	18.38
	3m	20.26	20.17	8.80
	6m	20.22	20.21	1.02
	1y	20.31	20.38	6.26
6m	1m	19.92	19.56	35.56
	3m	20.05	19.79	26.13
	6m	20.20	20.02	17.90
	1y	20.27	20.22	5.11
1y	1m	19.05	19.29	23.66
	3m	19.43	19.46	3.56
	6m	19.54	19.71	17.14
	1y	19.81	20.18	37.55

Parameters obtained from naïve calibration:

$$\sqrt{v_0} = 22.4\%$$

$$\kappa = 1.78$$

$$\sqrt{\theta} = 21.7\%$$

$$\eta = 44.65\%$$

$$\rho = -0.51$$

Cliquet in the Heston model

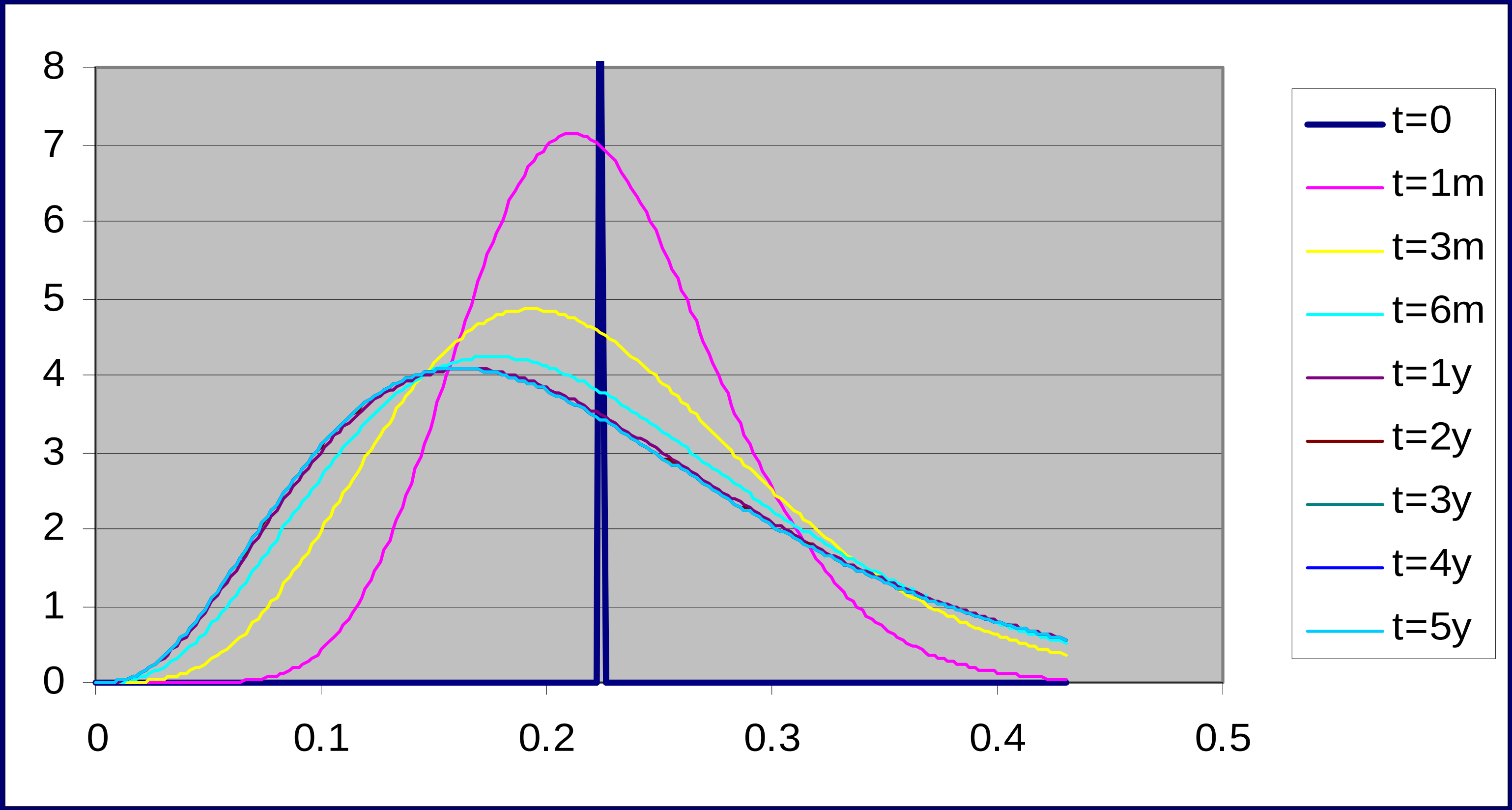
- Statistical analysis
  - For $t<T$, a price depending on S_T/S_t is determined by both $\int_t^T \sigma_s^2 ds$ and on the value of the volatility atm time t, σ_t .
  - Pricing a forward-start option at time 0 involves the joint distribution of

$$\left(\sigma_t, \int_t^T \sigma_s^2 ds\right)$$

  - The forward implied volatility surface of a model is sometimes dramatically different from the spot implied volatility surface.

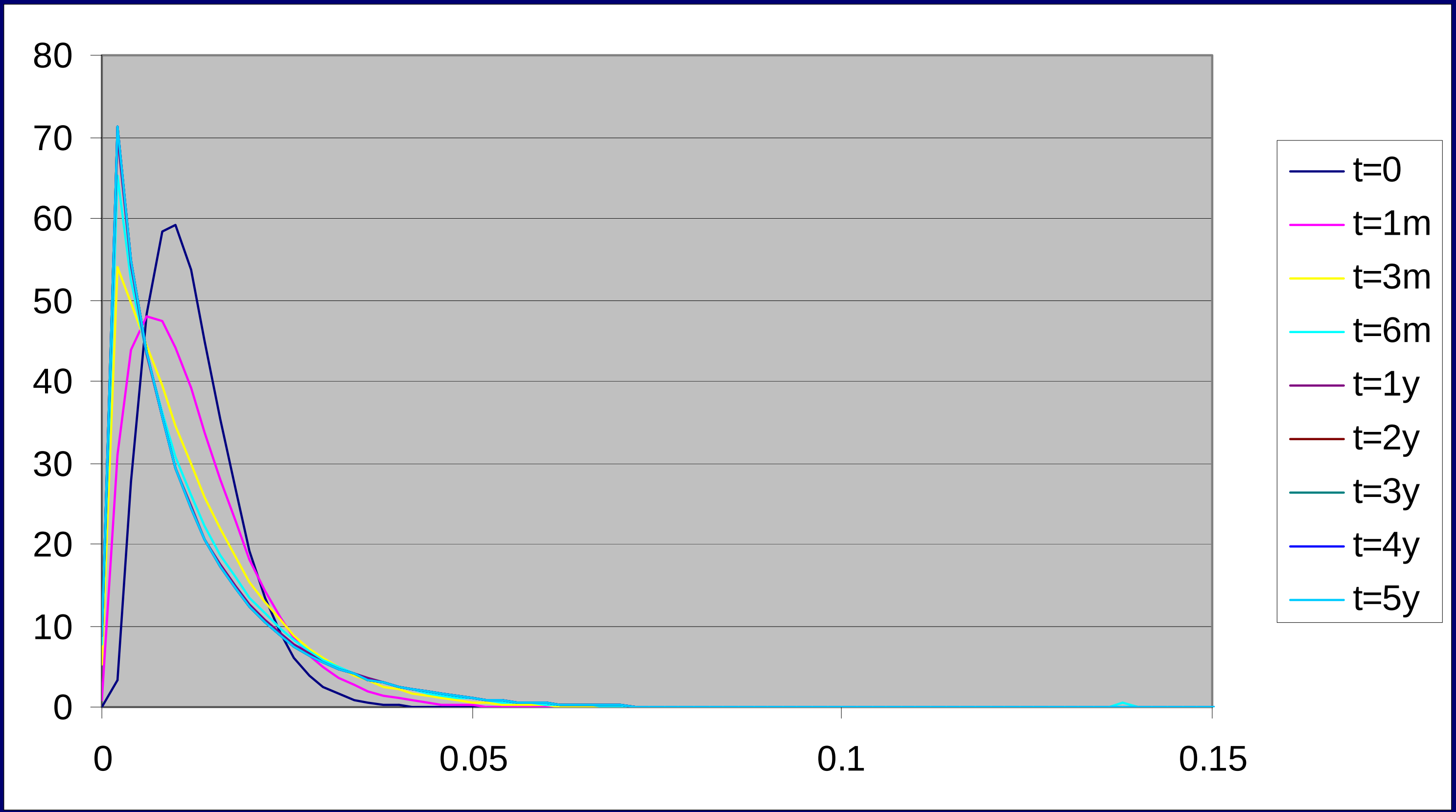
Statistical properties of variables

- Density function of σ_t



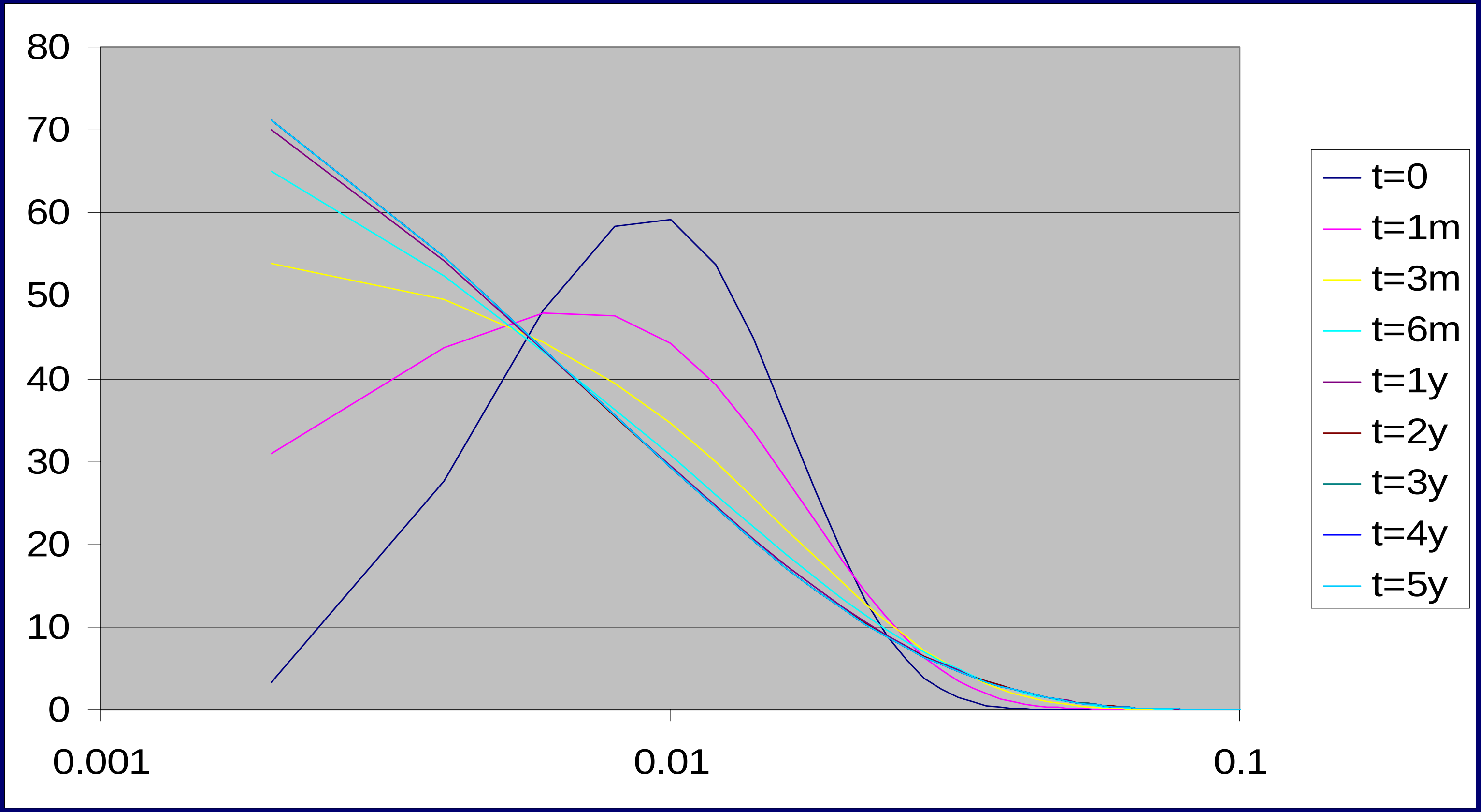
Implied forward volatility

- Density of $\int_t^{t+T} \sigma_s^2 ds$ in the Heston model. T=3 months.



Statistical properties

- Density of $\int_t^{t+T} \sigma_s^2 ds$, logarithmic scale

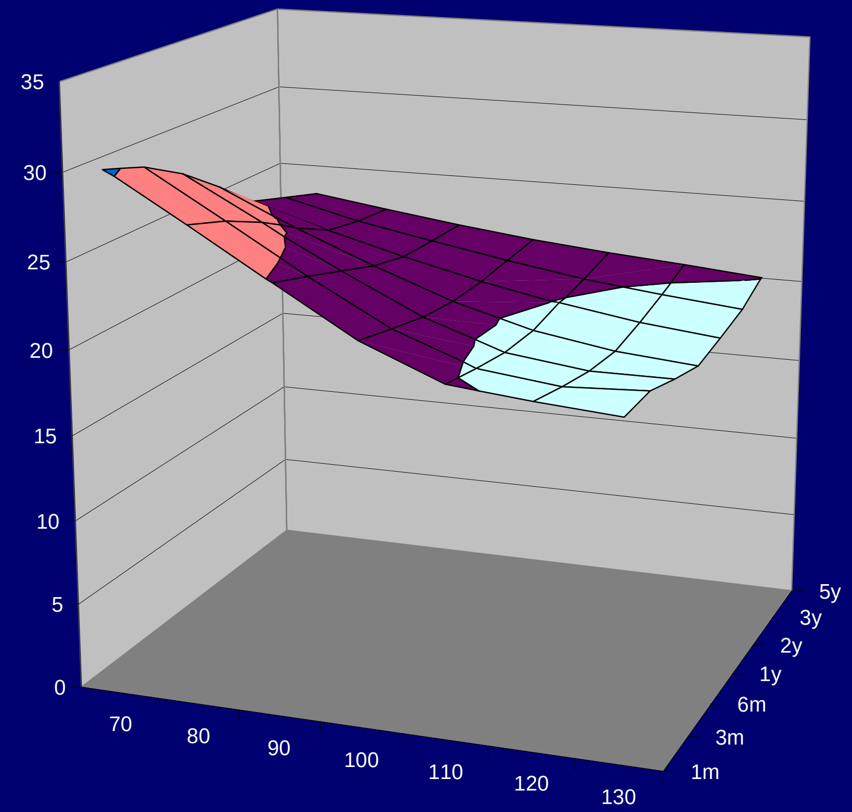


Statistical properties

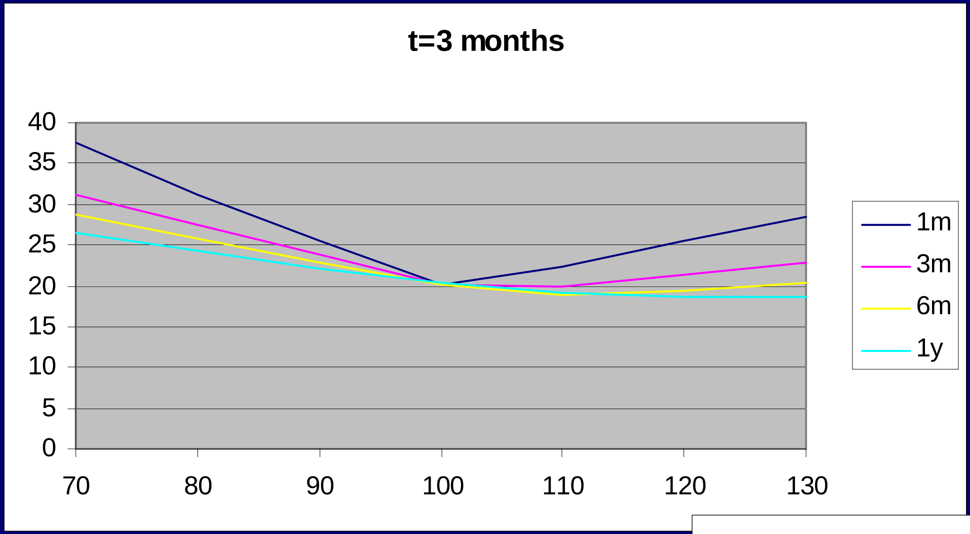
- When $t \neq 0$ the statistical properties of the relevant variables differ significantly from the case $t=0$.
- Hence, although the *static* properties of the model are good (I.e. fit to the implied vol surface), its *dynamic* properties are questionable.
- This is partly due to the presence of a hidden parameter: the “short volatility”.

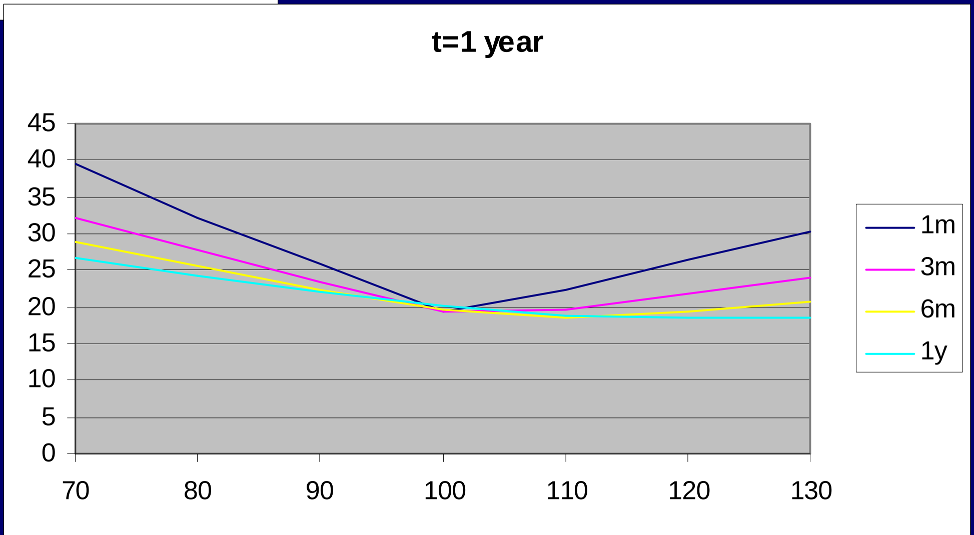
Heston implied volatilities

- Calibrated to the STOXX 50 (no penalization)

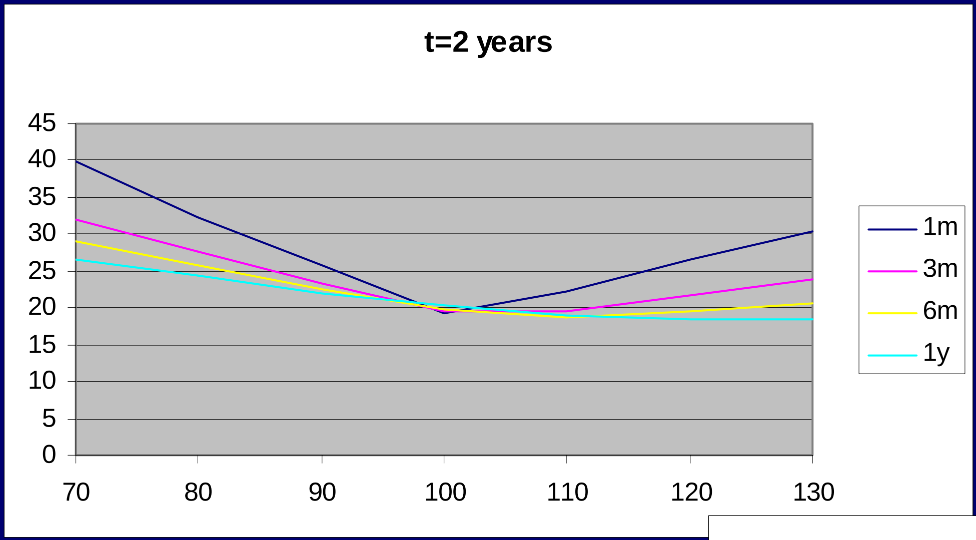


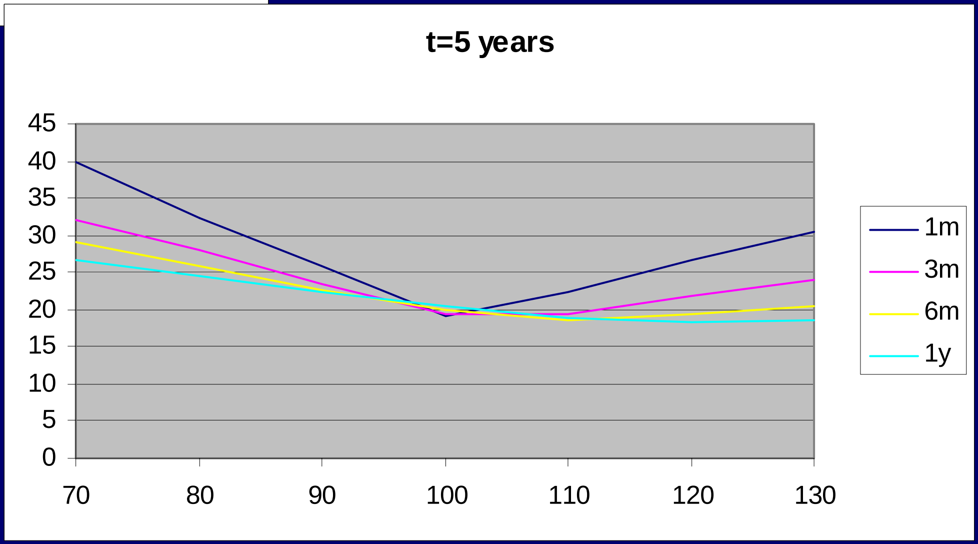
Heston implied forward vol





Heston implied forward volatility





Heston implied forward volatility

- The skew structure of implied volatilities for short maturities is affected by the forwarding.
- Short skew becomes U-shaped - this is inconsistent with observations.

Cliquets in the Heston model

- The change of shape in forward implied volatilies relative to spot implied volatilities can be used as a penalization for the calibration of the model.
- Once the model is properly calibrated, the hedging strategy for the cliquet can be computed as described earlier.

Further research

- More models are being investigated with the aim both to replicate the spot volatility surface and output consistent forward implied volatilities.
  - inhomogeneous Levy processes are investigated since they provide a stochastic quadratic variation
  - homogeneous processes with non-independent increments preserve the structure of the implied volatility surface under forwarding. They are difficult to handle in general as they are long-memory processes.

References

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- Schobel and Zhu, 1998: *Stochastic volatility with an Ornstein-Uhlenbeck process: an extension*. Working paper
- Most of the topics presented are reviewed in:
  Equity derivatives: Theory and applications, Wiley 2002