

Delta Hedging Works

On Market Completeness for Diffusion Processes.

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Abstract

We discuss market completeness for diffusion-driven financial models beyond the classic requirement that the volatility matrix of traded instruments is invertible: we show that the market of “relevant payoffs” generated by a finite-dimensional diffusion model

$$dX_t = \mu(X_t) dt + \sum_{j=1}^d \sigma^j(X_t) dW_t^j, \quad X_0 \in \mathbb{R}^m \quad (1)$$

is complete as soon as the diffusion *allows weak delta-hedging* in the sense that

$$h(t; x) := \mathbb{E} \left[\mathbb{H}(X_t) \mid X_0 = x \right]$$

is $d\langle X \rangle \otimes d\mathbb{P}$ -almost surely differentiable in x for all smooth non-negative \mathbb{H} with compact support (i.e. “delta” exists in a weak sense).

Completeness means that for all “relevant” payoffs $H_T \in L_+^1(\mathcal{F}_T^X)$ which are measurable with respect to X to X , there exists a \mathbb{F}^X -admissible process Δ such that

$$H_T = \mathbb{E} [H_T] + \sum_{i=1}^m \int_0^T \Delta_t^i d\tilde{S}_t^i \quad \text{with} \quad \tilde{S}_t^i := X_t^i - \int_0^t \mu_i(X_u) du. \quad (2)$$

The intuition is that if the value function has a weak concept of a “delta” for the very smoothest payoffs, then it is possible to approximate any payoff by hedgeable smooth functions. As an application, it is shown that the former is the case if the coefficients μ and σ are C^1 .

The condition to “allow weak delta-hedging” is much weaker than classic arguments for representation theorems which require invertibility of the volatility matrix σ .

Keywords: Complete Market, Diffusion, Predictable Representation Property, Extremal Martingale, Second fundamental theorem of asset pricing

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1 Introduction

The current article focuses on a classic question in financial mathematics:¹ whether a market described by a given financial model is *complete*. In financial terms, this means that every contingent claim in the market can be replicated by continuous hedging in the underlying tradable assets. In other words, completeness means that it is possible to hedge any risk arising from trading contingent claims by trading in liquid instruments. The equivalent mathematical statement is that the local martingales representing the (suitably normalized) liquid tradable instruments have the *predictable representation property*, or *PRP* (cf. Revuz/Yor [9]): every local martingale can be represented as a stochastic integral of over the local martingales of the discounted price processes.

A classical example is the *Black&Scholes* model [2]. It is given under the risk-neutral measure as the solution to the SDE

$$\begin{cases} dS_t^0 &= S_t^0 r_t dt \\ dS_t^1 &= S_t^1 \mu_t dt + S_t^1 \sigma dW_t \end{cases} \quad (3)$$

where $r = (r_t)_t$ denotes the risk-free deterministic interest rate, $\mu = (\mu_t)_t$ the deterministic yield received from holding the share and $\sigma > 0$ the *volatility* of the stock price process S^1 . The process S^1 is driven by the Brownian motion W which lives on a stochastic base (\mathbb{P}, \mathbb{F}) . The process S^0 , on the other hand, is the cash account.

If we continuously forward-sell all proceeds from the share over every time interval, then we effectively trading in the *forward* of the share,

$$\tilde{S}_t^1 := S_t^1 - \int_0^t S_u^1 \mu_u du$$

which is a martingale under the risk-neutral measure.² The corresponding process \tilde{S}^0 for the cash bond is constant $\tilde{S}^0 \equiv 1$.

A *contingent claim*, or *payoff*, is basically a non-negative integrable random variable $H_T \in L_+^1(\mathcal{F}_T)$ where $T < \infty$ is a stopping time which is uniformly

¹ Many thanks to Alexander Schied for his help all along. Any mistakes in this article are of course mine.

²Basically, this is a generalized concept of a “discounted stock price” since it also takes into account proceeds from dividends and stock borrow activity.

bounded away from infinity; we use the subscript T of a contingent claim H_T to refer to there relevant stopping time.³ A *market* \mathbb{M} is a set of such contingent claims. We then say that \mathbb{M} is *complete with respect to S over the filtration \mathbb{F}* if we can find for every $H_T \in \mathbb{M}$ an \mathbb{F} -predictable admissible integrand Δ for \tilde{S} such that

$$H_T = \mathbb{E} [H_T] + \int_0^T \Delta_u d\tilde{S}_u$$

(in this case the integral with respect to $d\tilde{S}^0$ vanishes).

It is well-know that Black&Scholes' model is complete for the market of all contingent claims $H_T \in L^1(\mathcal{F}_T)$ if \mathbb{F} is generated by W . Moreover, if $H_T = \mathbb{H}(S_T)$ and if

$$h(t; s) := \mathbb{E} [\mathbb{H}(S_T) \mid S_t = s]$$

is $C^{1,2}$, then we can identify the process Δ as

$$\Delta_t = \partial_s h(t; S_t) .$$

This derivative of the value function h in direction to a hedging instrument S is called the “delta” of the payoff H_T with respect to S . We say “delta-hedging works” in Black&Scholes' model.

The Case for Market Completeness

The motivation for the article comes from the observation that market completeness is only rarely covered in publications about financial modeling. Basically, when developing a traditional financial model based on the efficient market hypothesis,⁴ such as a stochastic volatility model or an interest model, not to mention models for pricing credit- or mortgage-linked products, two key properties should be satisfied:

- **Absence of Arbitrage**

Most importantly, the model has to be arbitrage-free, since otherwise, it would be possible to construct non-negative payoffs which trade at zero or less initial value; or negative payoffs with positive current value. For an institution which uses such a model to compute prices for outstanding derivatives positions, this means in reverse that even the *sign* of the trading book value might be incorrect.

- **Market Completeness**

The second key property of a traditional efficient-market based financial model is that it should at least theoretically allow to hedge the risks in a given payoff with the assumed liquid hedging instruments. For complex models, for example those in the recent stream of “infinite dimensional”

³Black&Scholes' model itself is not uniformly integrable; in particular, we have the pathwise limit $\lim_{t \uparrow \infty} S_t/B_t = 0$ almost surely.

⁴excluding models addressing market inefficiencies such as transaction costs, imperfect hedging, jumps, etc

curve models, this is clearly a non-trivial question. But it is also not a trivial question for n -factor stochastic volatility such as those discussed by Buehler [4].

The first requirement, absence of arbitrage, is usually addressed by modeling the model directly under a equivalent local martingale measure or by stating that a respective change from the natural into a risk-neutral local martingale measure is possible. The existence of such a measure is equivalent to the (stronger) “no free lunch without vanishing risk”, as has been shown by Delbaen/Schachermayer [5] in 1994 in the “first fundamental theorem of asset pricing”. Their recent book [6] provides a comprehensive overview over the spirit of the theory.

The second requirement of market completeness, however, is rarely discussed for a given model and usually implicitly assumed. Take, for example, a model for a stock price

$$dS_t = S_t \sigma_t dW_t$$

where σ is a predictable and suitable integrable process. The usual course of action is to invert the relation $d\tilde{S}_t = \sigma_t \tilde{S}_t dW_t$ to solve for dW_t as

$$dW_t = \frac{d\tilde{S}_t}{\sigma_t \tilde{S}_t} . \tag{4}$$

Since W itself has the predictable representation property, such a relation implies that \tilde{S} has it, too.⁵ This is discussed in several text books, for example in Shreve [11] page 232 under the topic “second fundamental theorem of asset pricing”. See also Karatzas/Shreve [10] and the references therein.⁶

However, we like to point out that (4) is a very strong condition since it aims to recover the background Brownian motion from observing the stock price. This, in turn, allows hedging all “payoffs” which depend on this background process. However, in practise, we are not really interested on contingent claims which depend on the artificial W , but rather on those depending on the asset prices S itself: payoffs such as $(S_T - K)^+$ for a simple call or $(K^2 - \frac{1}{T} \langle \log S \rangle_T)^+$ for a put on the realized quadratic variation of the returns of S are explicit functionals of the stock price path. We call such payoffs *relevant*.

What else would you be able to trade in a real market ?

Completeness of the Market of Relevant Payoffs

This article aims at contributing to the subject of market completeness in two ways: first of all, we argue that “market completeness” should be replaced by the more meaningful concept of “completeness of the market of relevant payoffs”, i.e. the ability to hedge payoffs whose value depend on the path of the value of tradable instruments. This is in contrast to general random variables which

⁵In this case, it is obvious to see that the local martingale measure for \tilde{S} must be unique.

⁶We basically need to make sure that (4) is a well-defined stochastic integral which is the case if $\int_0^t \sigma_t^{-2} dt < \infty$ which can be archived, for example, by bounding σ away from zero.

are measurable with respect to, say, the (unobservable) background driving Brownian motion.

Secondly, we provide a key theorem and some simple tools to help showing whether a given model is indeed complete for the relevant payoffs, and apply the provided tools to the concept of “infinite dimensional” models and their finite-dimensional parametrizations. In particular, we do *not* require invertibility of the volatility matrix of local martingale vector representing the tradable instruments. We also argue that this is not a natural condition anyway since volatility matrices of reduced rank arise naturally in many applications, in particular when modelling equity price processes in actual business times across different time zones and countries.

Delta-Hedging Works

The key property for market completeness is *weak delta-hedging*.

A stationary process S is said to allow *weak delta-hedging* if for all smooth non-negative functions \mathbb{H} with compact support the value function

$$h(t; s) := \mathbb{E} [\mathbb{H}(S_t) \mid S_0 = s]$$

has a “delta”, i.e. a weak derivative, “in the direction where the vector S has volatility”. This basically means that the payoff needs to be somehow differentiable in the value of each tradable instrument whose value may change over the next small interval of time. The idea is that we do not need to hedge tradable instruments which cannot move because the volatility is zero (for example if there is a holiday or a market disruption).

With proposition 2.2 we show that this is for example the case if the coefficients of the underlying diffusion process are C^1 .

Note that we in no way require invertibility of the volatility matrix σ .

Our main theorem 2.1 states then that if our model allows *weak delta-hedging* then the *market of relevant payoffs* is complete: all contingent claims H_T which depend on the filtration generated by S can be hedged in the sense that

$$H_T = \mathbb{E} [H_T] + \int_0^T \Delta_u d\tilde{S}_u$$

where Δ is an admissible integrand for S which is measurable with respect to the filtration generated by S , i.e. which is known from observing S .

Moreover, if the price process

$$H_t := \mathbb{E} [H_T \mid \mathcal{F}_t]$$

has the form $H_t = h(t; S_t)$ and is differentiable in S , then

$$\Delta_t = \partial_s h(t; S_t) .$$

In other words: *delta-hedging works*.

Beyond Finance

In non-financial Stochastic Analysis, the question of market completeness is equivalent to the question of “extremality” or of the “predictable representation property”.

Indeed, we will formulate our results in a more general setting of a diffusion process X as defined in equation (1) in the abstract. The question is when X is extremal on its own filtration (ie, has the predictable representation property on its own filtration). The answer is, as for market completeness, that this is the case if X allows *weak delta-hedging*, i.e. if

$$h(t; x) := \mathbb{E} [\mathbb{H}(X_t) \mid X_0 = x]$$

is $d\langle X \rangle \otimes dP$ -almost surely differentiable in x . In that case, we can write

$$H_T = H_0 + \sum_{i=1}^m \int_0^T \Delta_t^i d\tilde{S}_t$$

for the associated martingale process defined in (2).

1.1 Notation

For most of the discussion we will adapt the notation of Revuz/Yor [9] including the short-hand notations $x \vee y := \max\{x, y\}$, $x \wedge y := \min\{x, y\}$, $x^+ := x \vee 0$ and $x^- := (-x)^+$.

As before, we are given a stochastic base $\mathbb{W} = (\Omega, \mathbb{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_t$ which satisfies the usual conditions and supports a d -dimensional standard Brownian motion W . We call a filtration $\mathbb{G} = (\mathcal{G}_t)_t$ a sub-filtration of \mathbb{F} , denoted by $\mathbb{G} \subseteq \mathbb{F}$, if $\mathcal{G}_t \subseteq \mathcal{F}_t$ for all t . For a stochastic process X on \mathbb{F} which is defined up to a positive stopping time τ , we denote by $\mathbb{F}^X = (\mathcal{F}_t^X)_t$ completed filtration generated by the stopped process $X^\tau = (X_t^\tau)_t$ with $X_t^\tau := X_{t \wedge \tau}$. We also use $\mathbb{F}^\tau = (\mathcal{F}_t^\tau)_t$ to denote the (larger) stopped version of \mathbb{F} , i.e. $\mathcal{F}_t^\tau := \mathcal{F}_{\tau \wedge t}$.

In terms of notation, we use $L^p(\mathcal{A})$ for a σ -algebra \mathcal{A} to denote the \mathcal{A} -measurable random variables H such that $\mathbb{E}[|H|^p] < \infty$. Its subset of non-negative variables is denoted by $L_+^p(\mathcal{A})$. The space of all continuous local martingales $X = (X_t)_t$ on a filtration \mathbb{G} is denoted by $\mathcal{M}^{\text{loc}}(\mathbb{G})$, while $\mathcal{M}(\mathbb{G})$ denotes all true martingales and $\mathcal{M}^2(\mathbb{G})$ all square-integrable martingales. We may use a subscript “+” to denote non-negative elements. We may also omit the notion of the filtration \mathbb{F} .

Let $X = (X^1, \dots, X^d)$ be a square-integrable martingale. We define the set of *admissible integrands* $\mathcal{H}^2(X, \mathbb{G})$ over a filtration $\mathbb{G} \subseteq \mathbb{F}^\tau$ as all \mathbb{G} -predictable processes $\varphi = (\varphi_t)_t$ such that

$$\mathbb{E} \left[\sum_{i=1}^m \int_0^{T \wedge \tau} \|\varphi_s^i\|_2^2 d\langle X^i \rangle_s \right] < \infty$$

such that

$$d\tilde{S}_t^i = \sum_{j=1}^d \sigma_i^j(X_t^\tau) dW_t^j \quad (6)$$

on $\{t < \tau\}$. Note that $\tilde{S} \in \mathcal{M}^{\text{loc}}(\mathbb{F}^X)$ and also the notational convenience that $\tilde{S}^\tau = \tilde{S}$ since \tilde{S} is stopped in τ . Financially, the martingale part corresponds to the instantaneous forward price return process and it can be realized by continuously entering into “instantaneous” forwards.

DEFINITION 2.1 (Replication) *Let $H_T \in L_+^1(\mathcal{F}_T)$ be a non-negative, integrable random variable with maturity T (i.e., it is measurable with respect to \mathcal{F}_T where $T < \tau$ is a stopping time).*

We say that H_T is replicable with respect to X (or \tilde{S}) over a filtration $\mathbb{G} = (\mathcal{G}_t)_t$ if there exists a vector process $\Delta \in \mathcal{H}^{\text{loc}}(\tilde{S}, \mathbb{G})$ such that

$$H_T = \mathbb{E}[H_T] + \sum_{i=1}^m \int_0^T \Delta_u^i d\tilde{S}_u^i. \quad (7)$$

We call the process $\Delta = (\Delta^1, \dots, \Delta^m)$ the “delta” of H_T .

EXAMPLE 1 *Let $H_T := \mathbb{H}(X_T)$ where $T < \tau$ and \mathbb{H} is a non-negative function such that $\mathbb{E}[H_T] < \infty$. Define the value function of \mathbb{H} as*

$$h(t; x) := \mathbb{E}[\mathbb{H}(X_T) \mid X_t^T = x]$$

and assume that h is in $C^{1,2}$. Then,

$$H_T = \mathbb{E}[H_T] + \sum_{i=1}^m \int_0^T \Delta_u^i d\tilde{S}_u^i \quad (8)$$

with the classic

$$\Delta_t^i := \partial_{x^i} h(t; X_t^T).$$

Proof – Use Ito and the martingale property of the process $(h(t; X_t^T))_t$. Note that if \tilde{S} is not bounded, then (8) holds only locally. \square

Extremality and the PRP

DEFINITION 2.2 (Extremality) *Let \mathbb{G} be a sub-filtration of \mathbb{F} . We say that X (or \tilde{S}) is extremal on \mathbb{G} or that it has the predictable representation property, PRP, on \mathbb{G} , iff all $H \in L_+^1(\mathcal{G}_\infty)$ are replicable with respect to X on \mathbb{G} .*

We say that X (or \tilde{S}) is naturally extremal if it is extremal on its own filtration \mathbb{F}^X .

It is worthwhile to note that extremality for random variables is equivalent to the ability to write any local martingale as an integral over the martingale part \tilde{S} of X :

PROPOSITION 2.1 *Let $\mathbb{G} \subseteq \mathbb{F}$ be a filtration. Then, X is extremal on \mathbb{G} if and only if for all non-negative local martingales $H = (H_t)_{t \leq T}$ on \mathbb{G} there is a process $\Delta \in \mathcal{H}^{\text{loc}}(\tilde{S}, \mathbb{G})$ such that*

$$H_t = H_0 + \sum_{i=1}^m \int_0^t \Delta_u^i d\tilde{S}_u^i . \quad (9)$$

Proof – Assume first that X is extremal and that H is a local martingale. Let then $(\tau_n)_n$ with $\tau_0 = 0$ be a localizing sequence for H such that H^{τ_n} is in \mathcal{M}^2 . Hence, we find $\Delta^{(n)} \in \mathcal{H}^{\text{loc}}(\tilde{S}, \mathbb{G})$ such that

$$H_T^{\tau_n} = H_0 + \sum_{i=1}^m \int_0^T \Delta_u^{(n),i} d\tilde{S}_u^i$$

Define then $\Delta_t := \Delta_t^{(n)}$ for $\tau_{n-1} \leq t < \tau_n$ which is the process in (9).

To prove the reverse for a given random variable H_T , simply define the martingale $H_t := \mathbb{E}[H_T | \mathcal{G}_t]$ and use its representation (9) to replicate H_T . \square

Delta-Hedging Works

Assume that $\mathbb{H} : \mathbb{R}^m \rightarrow \mathbb{R}_+$ is a non-negative function and that $T < \tau$ is a stopping time such that $\mathbb{E}[\mathbb{H}(X_T)] < \infty$. As before, we call

$$h(t; x) := \mathbb{E}[\mathbb{H}(X_T) | X_t^T = x] \quad (10)$$

the *value function* of \mathbb{H} . Basically, we want to prove two statements in this article:

The first is that as soon as $h(t; \cdot)$ is almost surely differentiable in the directions of x where it is actually diffusive, then we can write

$$\mathbb{H}(X_T) = \mathbb{E}[\mathbb{H}(X_T)] + \sum_{i=1}^m \int_0^T \Delta_t^i d\tilde{S}_t^i \quad (11)$$

with

$$\Delta_t^i := \partial_{x^i} h(t, X_t^T) .$$

This is the classic “delta-hedging” idea in mathematical finance. As shown with example 1, this is obviously the case if $h(t; \cdot)$ is C^2 , but we show the stronger case for only almost sure differentiability in x .⁷ The condition that h only needs to be differentiable “in the direction where X is actually diffuse” intuitively means that we do not need to hedge our exposure to X^i if its volatility for the next increment dt vanishes, i.e. if locally $\langle X^i \rangle|_{[t, t+dt]} \equiv 0$.

For example, the following formulation represents an extended Black&Scholes-model with time-dependent drift m and volatility Σ :

$$\begin{cases} dX_t^1 &= X_t^1 m(X_t^2) dt + X_t^1 \Sigma(X_t^2) dW_t^1 \\ dX_t^2 &= dt \end{cases}$$

⁷This covers the case where h is absolutely continuous in x .

with $X_0^1 = \tilde{S}_0$ and $X_0^2 = 0$ (note that $X_t^2 = t$). In this case, we expect to be able to replicate a payoff just if it is differentiable with respect to X^1 – differentiability in direction to $X_t^2 = t$ is not really required.

In mathematical terms, the appropriate formulation of this condition is to say that h is almost surely differentiable with respect to the product measure $d\langle X^\tau \rangle \otimes d\mathbb{P}$, i.e.

$$\int \int 1_{\partial_x|_{x=X_t(\omega)} h(t;x)} \text{ does not exist or is not continuous } d\langle X^\tau \rangle_t(\omega) d\mathbb{P}(\omega) = 0 .$$

The second key statement is that if such an expression exists for all smooth functions \mathbb{H} with compact support, then by extension we can represent *any* measurable variable $H_T \in L^1(\mathcal{F}_T^X)$ as

$$H_T = \mathbb{E}[H_T] + \sum_{i=1}^m \int_0^T \Delta_t^i d\tilde{S}_t^i$$

for an admissible “delta” $\Delta \in \mathcal{H}^{\text{loc}}(\tilde{S}, \mathbb{F}^X)$. In financial theory, this means that the market of all payoffs depending on the tradable instruments $\tilde{S} = (\tilde{S}^1, \dots, \tilde{S}^m)$ is complete.

This idea is formalized in the following definition:

DEFINITION 2.3 (Weak Delta-Hedging) *Define the operator*

$$\mathbb{P}f_t(x) := \mathbb{E} [f(X_t^\tau) \mid X_0 = x] .$$

*We say that X allows weak delta-hedging if $\mathbb{P}f_t(\cdot)$ is $d\langle X^\tau \rangle \otimes d\mathbb{P}$ -almost surely differentiable for all $f \in C_K^\infty$, i.e. for all smooth functions with compact support.*⁸

Note that this property was called *weak preservation of smoothness* in [3].

THEOREM 2.1 (“Delta Hedging Works”) *If X allows weak delta-hedging, it is naturally extremal.*

Moreover, if a local martingale $H = (H_t)_{t \leq T}$ with $T < \tau$ is given as $H_t = h(t; X_t^T)$ in terms of a function $h(t; \cdot)$ which is $d\langle X^\tau \rangle \otimes d\mathbb{P}$ -almost surely differentiable in x and continuous in t , then the delta in equation (9) is given as the natural

$$\Delta_t^i := \partial_{x^i} h(t; X_t^T) . \tag{12}$$

The following proposition provides a good handle for “weak delta-hedging”. In financial terms, it states that if drift and volatility in equation (5) are C^1 , then the market of “relevant payoffs”, i.e. those depending on the observable X , is complete.

⁸Note that this implies that if $\langle \tilde{S} \rangle$ is absolutely continuous with respect to the Lebesgue measure, then the condition is satisfied if $\mathbb{P}f_t$ is absolutely continuous.

PROPOSITION 2.2 *If μ and σ in equation (5) are differentiable with locally Lipschitz derivatives, then X allows weak delta-hedging and is therefore naturally extremal.*

Proof – To highlight the dependency of X on its origin, $x := X_0$, we now write X^x . Under the assumptions of the proposition, theorem 39 in Protter [8], pg. 305 shows that the map

$$x \longmapsto X_t^x(\omega)$$

is in C^1 for almost all (t, ω) , and there exists a continuous “derivative” process $Z^{(x,i)} = (Z^{(x,i),1}, \dots, Z^{(x,i),m})$ for all $i = 1, \dots, m$ such that $Z_t^{(x,i)} = \partial_{x^i} X_t^x$ almost surely. The vector $Z_t^{(x,i)} = (Z_t^{(x,i),1}, \dots, Z_t^{(x,i),m})$ satisfies

$$dZ_t^{(x,i)} = \sum_{k=1}^m Z_t^{(x,i),k} \left\{ (\partial_{x^k} \mu)(X_t) dt + \sum_{j=1}^d (\partial_{x^k} \sigma^j)(X_t) dW_t^j \right\} \quad (13)$$

and is defined up to the same explosion time τ as X^x .⁹ Note that the derivative process is positive.

Let now $\mathbb{H} \geq 0$ be some C_K^∞ function and fix some $t > 0$. Since $\partial_{x^i} \mathbb{H}$ has compact support, say on $D \subseteq \mathbb{R}^m$, its support is bounded, hence the continuous function $\mathbb{H}' := \partial_{x^i} \mathbb{H}$ is bounded by the constant $K = \|\mathbb{H}\|_\infty$, independently of x . Moreover, it vanishes outside D . Continuity in t follows from the Feller property of the diffusion X .

From the positivity of $Z_t^{(x,i)} \mathbb{H}(X_t^x)$ it follows that

$$\partial_{x^i} \mathbb{E}[\mathbb{H}(X_t^x)] = \mathbb{E}\left[Z_t^{(x,i)} \mathbb{H}(X_t^x)\right],$$

Lebesgue-almost surely.¹⁰ Since $d\langle \tilde{S}^i \rangle_t = \sum_j \sigma_i^j(X_t)^2 dt \ll dt$, we have proven that X weakly preserves smoothness. \square

Before we proceed the proof of theorem 2.1 on page 13, let us briefly comment on classic ways of dealing with replicability:

⁹To see that (13) does not explode before τ , fix x and let $\tau_n := \inf\{t : \|X_t\| \geq n\} \wedge \tau$. Then, the stopped processes X^n are bounded and so are the coefficients on the right hand side of (13), i.e. up to each τ_n , $z \mapsto z \partial_{x^i} \mu(X_t)$ and $z \mapsto z \partial_{x^i} \sigma^j(X_t)$ are globally Lipschitz, which implies that a solution for (13) exists up to each τ_n . Taking the limit yields that $Z^{(i)}$ is well-defined up to τ . See the proof of theorem 39 pg. 305 in Protter [8] for more details.

¹⁰To see this, fix some $x_0^i < x^i$ such that $x_0 := (x^1, \dots, x^{i-1}, x_0^i, x^{i+1}, \dots, x^m)$ is in the domain of X_0 . Let $\mathcal{H}(y^i) := \mathbb{E}[\int_{x_0^i}^{y^i} Z_t^{(z,i)} \mathbb{H}(X_t^z) dz^i] + \mathbb{E}[\mathbb{H}(X_t^{x_0})]$ where we set $z := (x^1, \dots, x^{i-1}, z^i, x^{i+1}, \dots, x^m)$ for ease of notation. Note that $\mathcal{H}(x^i) = \mathbb{E}[\mathbb{H}(X_t^x)]$.

Tolleni’s theorem allows us to exchange integration such that $\mathcal{H}(y^i) = \int_{x_0^i}^{y^i} \mathbb{E}[Z_t^{(z,i)} \mathbb{H}(X_t^z)] dz^i + \mathbb{E}[\mathbb{H}(X_t^{x_0})]$. Hence, \mathcal{H} is absolutely continuous with respect to the Lebesgue measure with derivative $\partial_{x^i} \mathbb{E}[\mathbb{H}(X_t^x)] = \mathbb{E}\left[Z_t^{(x,i)} \mathbb{H}(X_t^x)\right]$.

Classic Approach via Inversion of the Volatility Matrix

The classic answer to the question of extremality and therefore market completeness in financial applications rests on two assumptions: the first is that the Brownian motion W is extremal on the filtration F^X , and the second is that the volatility matrix σ is sufficiently invertible.

The first condition, extremality of W , means that for each $H_T \in L_+^1(\mathcal{F}_T^X)$ there is an admissible process $\xi \in \mathcal{H}^{\text{loc}}(W, \mathbb{F}^X)$ such that

$$H_T = \mathbb{E}[H_0] + \sum_{j=1}^d \int_0^T \xi_t^j dW_t^j .$$

In the light of equation (6) on page 8, invertibility of the volatility matrix basically means that we can write

$$dW_t = \sigma^{-1}(X_t) d\tilde{S}_t , \quad (14)$$

in which case $\Delta_t := \xi_t \cdot \sigma(X_t)^{-1}$ satisfies the replication equation (7).

An example for the aforementioned invertibility conditions is given by Janson/Tysk [7] (theorem 2.7). Other examples outside the context of SDEs, i.e. for general integrals can be found in Shreve [11] and Karatzas-Shreve [10].

THEOREM 2.2 *Assume that W is extremal on \mathbb{F} , that $m \geq d$ and that the local martingale $\tilde{S} = (\tilde{S}^1, \dots, \tilde{S}^d)$ solves the SDE*

$$d\tilde{S}_t^i = \sum_{j=1}^d \Sigma_i^j(t; \tilde{S}_t) dW_t^j$$

with a continuous and locally Lipschitz matrix $\Sigma(t; \cdot)$ which has rank d for all t . Also assume that $\|\Sigma(t; s)\| \leq D(1 + \|s\|)$ on $[0, \tau] \times \mathbb{R}^d$ for some constant D .

Then, $h_t(s) := \mathbb{E} \left[\mathbb{H}(\tilde{S}_t) \mid \tilde{S}_0 = s \right]$ is in C^2 for all \mathbb{H} with $\mathbb{H}(\tilde{S}_T) \in L_+^1(\mathcal{F}_{T^}^{\tilde{S}})$.*

In other words, \tilde{S} weakly preserves smoothness.

However, the requirement of being able to recover the background Brownian motion W is much more demanding than our approach: first of all, it requires that the number of Brownian motions is less or equal as the dimensionality of the (diffusive parts of) the vector X . This is certainly a strong constraint, in particular when we come to the discussion of market completeness of factor models.

Moreover, invertibility is also not an intuitive restriction: there is no genuine reason why a covariance matrix for financial data should not be singular. One example are baskets of stocks from different countries, each of which may have different public holidays.

2.2 Proof of Theorem 2.1

In this section we assume that X weakly preserves smoothness as in definition 2.3. Before we begin, recall the following basic concept from distribution theory:

DEFINITION 2.4 A Dirac-Sequence is a sequence $(\varphi_n)_n$ of non-negative smooth $L^\infty(\lambda^m)$ -functions $\varphi_n : \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}$ with compact support and $\int_{\mathbb{R}^m} \varphi_n(x) dx = 1$ which converge against the Dirac measure in the sense of distributions – i.e. for any measurable f and

$$f_n(x) := \int_{\mathbb{R}^m} f(y) \varphi_n(y - x) dy$$

we have

$$f_n(x) \xrightarrow{L^p(\lambda_m)} f(x)$$

for $1 \leq p < \infty$.

EXAMPLE 2 An example for a Dirac sequence is

$$\varphi_n(x) := \frac{\Psi(xn)}{\int_{\mathbb{R}} \Psi(yn) dy}$$

with

$$\Psi(x) := 1_{|x| < 1} e^{-\frac{1}{1-x^2}}$$

PROPOSITION 2.3 Let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be a Lebesgue-measurable function and $(\varphi_n)_n$ a Dirac-sequence and f_n as above. Then,

- Each f_n is smooth.
- The derivatives of f_n converge to those of f wherever they exist.
- If f is bounded, then $f_n \rightarrow f$ in $L^\infty(\lambda_m)$, too.

Proof Outline

The main idea of the following proof for theorem 2.1 is as follows (with a bit of loose notation):

- We initially restrict ourselves to payoff functions of the class of smooth functions with compact support. For such a function \mathbb{H} , we define the value functions $h(t; x) := \mathbb{E}[\mathbb{H}(X_T) \mid X_t = x]$. To make things easier to control, we also assume that X and therefore \tilde{S} are bounded.

- The first complication arises because we only know that $h(t; \cdot)$ is C^1 and that it is continuous in t (due to the Feller property of X). If h were C^2 in x and C^1 in t , then we could simply apply Ito and would get that

$$dh(t; X_t) = \sum_{i=1}^m \partial_{x^i} h(t; X_t) d\tilde{S}_t^i \quad (15)$$

due to the martingale property of $H_t := h(t; X_t)$. Hence, the task at this point is to show that (15) also holds if we only have C^1 value functions.

- The second step is basically to approximate arbitrary measurable random variables by smooth functions with compact support.
- The final step is to generalize to the general case where \tilde{S} is merely a local martingale.

Step 1: C^1 -differentiability is sufficient

LEMMA 2.1 *Assume that X is bounded and that it allows weak delta-hedging. Let $H_T = \mathbb{H}(X_T)$ for $T < \tau$ and $\mathbb{H} \in C_K^\infty$ and set $h(t; x) := \mathbb{E}[\mathbb{H}(X_T) | X_t = x]$.*

Then,

$$H_T = \mathbb{E}[H_T] + \sum_{i=1}^m \int_0^T \Delta_t^i d\tilde{S}_t^i$$

with $\Delta_t^i := \partial_{x^i} h(t; X_t)$.

Proof – For $t \leq T$ define the value process

$$H_t := h(t; X_t) = \mathbb{E}[\mathbb{H}(X_T) | X_t] .$$

Since \mathbb{H} has compact support and is continuous, it is bounded and so is H . Hence, $(H_t)_t$ is a non-negative and bounded martingale. Recall that $h(t; \cdot)$ is $d\langle X \rangle \otimes d\mathbb{P}$ -almost surely differentiable by assumption.

The aim is to show that we can approximate the process $(h(t; X_t))_t$ with a sequence of processes $(h_n(t; X_t))_t$ with smooth h_n , and then to show that the processes and their representations converge in $L^2(\mathbb{P})$ to an element $\Delta \in L^{\text{loc}}(\tilde{S})$.

(a): Let us choose a “Dirac sequence” as defined above. Define

$$h_n(t; x) := (\varphi_n \star h)(t; x) := \int_{\mathbb{R}^m} \int_0^T h(t'; x') \varphi_n(t - t', x - x') dt' dx' . \quad (16)$$

For each $n \in \mathbb{N}$, the function h_n is smooth in all $(m + 1)$ parameters and we have $h_n \rightarrow h$ and $\partial_{s^k} h_n \rightarrow \partial_{s^k} h$ as $n \uparrow \infty$ in $L^p(\lambda^m)$ for all $p \in [1, \infty]$ following proposition 2.3.

In particular, we get that $h_n \rightarrow h$ in $L^\infty(\lambda^m)$ which implies that the sequence $h_n(t; X_t)$ converges to the bounded martingale $h(t; X_t)$ as $n \uparrow \infty$ in $L^2(\mathbb{P})$:

$$\mathbb{E} \left[(h_n(t; X_t) - h(t; X_t))^2 \right] \leq \|h_n - h\|_\infty^2 \downarrow 0 .$$

(b): Next, note that for each n , Ito can be applied. Since X is bounded, this yields in L^2 (not only locally)

$$\begin{aligned} h_n(t; X_t) - h_n(0; X_0) &= \sum_{i=1}^m \int_0^t \partial_{x^i} h_n(u; X_u) d\tilde{S}_u^i \\ &\quad + \int_0^t \partial_t h_n(u; X_u) du \\ &\quad + \frac{1}{2} \sum_{i,k=1}^m \int_0^t \partial_{x^i x^k} h_n(u; X_u) d\langle \tilde{S}^i, \tilde{S}^k \rangle_u . \end{aligned}$$

In step 1, we have shown that the left hand side converges in $L^2(\mathbb{P})$ against the L^2 -martingale $h(t; X_t) - h(0; X_0)$. This means that the drift terms on the right hand side of the above equation must vanish. As a result we get

$$h(t; X_t) - h(0; X_0) = \lim_{n \uparrow \infty} \sum_{i=1}^m \int_0^t \partial_{x^i} h_n(u; X_u) d\tilde{S}_u^i ,$$

where the limit is taken in $L^2(\mathbb{P})$.

(c): It remains to show that the hedging ratios $\partial_{x^i} h_n(t; X_t)$ converge against the hedging strategy Δ with $\Delta_t^i := \partial_{x^i} h_n(t; X_t)$ and that it is actually an element of $L^{\text{loc}}(\tilde{S})$.

To this end, let $\tau_\ell := \inf\{t : \|X\tau_t\|_2^2 > \ell\} \wedge \tau$ for $\ell \in \mathbb{N}$. On $\{t \leq \tau_\ell\}$, the process X is bounded and we get

$$\begin{aligned} &\mathbb{E} \left[\int_0^{\tau_\ell \wedge T} \left(\partial_{x^i} h_n(u; X_u) - \partial_{x^i} h(u; X_u) \right)^2 d\langle \tilde{S}^i \rangle_u \right] \\ &\leq \mathbb{E} \left[\int_0^{\tau_\ell \wedge T} \left(\sup_{t,x: \|x\|_2^2 \leq \ell} |\partial_{x^i} h_n(t; x) - \partial_{x^i} h(t; x)| \right)^2 d\langle \tilde{S}^i \rangle_u \right] \longrightarrow 0 \quad (\ell \uparrow \infty) . \end{aligned}$$

(note that these equations require that $\partial_{x^i} h(u; X_u)$ exists only $d\langle X \rangle \otimes d\mathbb{P}$ -almost surely). The result simply means that

$$\partial_x h_n(u; X_u) \xrightarrow{d\langle X \rangle \otimes d\mathbb{P}} \partial_x h(t; X_t)$$

such that $\Delta \in L^{\text{loc}}(\tilde{S})$, as required. Moreover, using our previous result that $h(t; X_t) \in L^2(\mathbb{P})$, we have

$$\mathbb{E} \left[\sum_{i=1}^m \int_0^t \Delta_t^{i2} d\langle \tilde{S}^i \rangle_t \right] = \mathbb{E} \left[\left(h(t; X_t) - h(0; X_0) \right)^2 \right] < \infty ,$$

i.e. $\Delta \in L^2(\tilde{S})$. Also note that non-negativity of the value process is guaranteed by construction. \square

Step 2: Generalization to arbitrary integrable random variables

LEMMA 2.2 *Assume that X is bounded and allows weak delta-hedging. Let $H_T = \mathbb{H}(X_T)$ for a non-negative, measurable function $\mathbb{H} : \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}$ and $T < \tau$.*

Then, H_T is replicable by X .

Moreover, if the value function $h(t; x)$ is a.s. differentiable in x , then $\Delta_t^i := \partial_{x^i} h(t; X_t)$.

Proof – We again use a Dirac-sequence, this time to smooth the payoff itself: let

$$\mathbb{H}^n(t; x) := (\varphi_n \star \mathbb{H})(t; x) := \int_{\mathbb{R}^m} \mathbb{H}(x') \varphi_n(x - x') dx' ,$$

then each \mathbb{H}^n is smooth with compact support.

According to the proof of our previous lemma 2.1, we can use

$$h^n(t; x) := \mathbb{E} [\mathbb{H}^n(X_T) \mid X_t := x]$$

to get in $L^2(\mathbb{P})$ the martingale representation

$$\mathbb{H}^n(X_T) = h^n(0; X_0) + \sum_{i=1}^n \int_0^T \partial_{x^i} h^n(t; X_t) d\tilde{S}_t^i .$$

Since $\mathbb{H}^n \rightarrow \mathbb{H}$ in $L^\infty(\lambda^n)$ on the left side, it follows as before that the right side converges in $L^2(\mathbb{P})$. Therefore, the vector $(\partial_{x^1} h^n(t; X_t), \dots, \partial_{x^m} h^n(t; X_t))$ converges as above in $L^2(\tilde{S})$ to some \mathbb{F}^X -predictable $\Delta \in L^2(\tilde{S}, \mathbb{R}^X)$ which replicates H_T .

If additionally the value function of \mathbb{H} is differentiable, then the hedging coefficient converge as before, too. \square

LEMMA 2.3 *Assume that X is bounded and allows weak delta-hedging. Let $H_T = \mathbb{H}(X_{T_1}, \dots, X_{T_n})$ with $T := T_n < \tau$ for a non-negative, measurable function $\mathbb{H} : \mathbb{R}^m \otimes \dots \otimes \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}$.*

Then, H_T is replicable by X .

Moreover, if the value function $h(t; x)$ is a.s. differentiable in x , then $\Delta_t^i := \partial_{x^i} h(t; X_t)$.

Proof – Assume w.l.g. that $T_{\ell-1} < t \leq T_\ell$. Then, the same procedure as in the proofs for the previous lemmata can be applied on $(T_{\ell-1}, T_\ell]$ to the measurable function

$$x \longmapsto \mathbb{E} [\mathbb{H}(X_{T_1}(\omega), \dots, X_{T_{\ell-1}}(\omega), X_{T_\ell}, \dots, X_{T_n}) \mid X_t = x] ,$$

conditional on $\mathcal{F}_{T_{\ell-1}}$. \square

LEMMA 2.4 *Assume that X is bounded and allows weak delta-hedging. Let H_T is a \mathcal{F}_T^X -measurable, integrable and non-negative random variable for some $T < \tau$.*

Then, H_T is replicable by X .

Moreover, if the value function $h(t; x)$ is a.s. differentiable in x , then $\Delta_t^i := \partial_{x^i} h(t; X_t)$.

Proof – Chose a countable representation $\{t_\ell\}_{\ell \in \mathbb{N}}$ of \mathbb{Q} . For all n , let $T_\ell^n := t_\ell$ for $\ell = 1, \dots, n$. Define a complete, discrete-time filtration $\mathbb{G} = (\mathcal{G}_n)_{n \in \mathbb{N}}$ by

$$\mathcal{G}_n := \sigma(X_{T_1^n}, \dots, X_{T_n^n}) \bigvee \mathcal{F}_0 .$$

As a next step, consider the discrete-time \mathbb{G} -martingale

$$H^n := \mathbb{E}[H_T | \mathcal{G}_n] = \mathbb{E}[H_T | X_{T_1^n}, \dots, X_{T_n^n}] .$$

For each of the random variables H^n lemma 2.3 yields a $\Delta^{(n)} = (\Delta^{(n),1}, \dots, \Delta^{(n),m}) \in L^2(\tilde{S})$ such that

$$H^n = \mathbb{E}[H^n] + \sum_{i=1}^m \int_0^T \Delta_t^{(n),i} d\tilde{S}_t^i .$$

in $L^2(\mathbb{P})$. The martingale $(H^n)_n$ converges in L^2 to H_T , hence $\Delta^{(n)}$ converges in $L^2(\langle \tilde{S} \rangle \otimes \mathbb{P})$ to some $\Delta \in L^2(\tilde{S})$ such that

$$H_T = \mathbb{E}[H_0] + \sum_{i=1}^m \int_0^T \Delta_t^i d\tilde{S}_t^i ,$$

as desired. The form $\Delta_t^i = \partial_{x^i} h(t; X_t)$ is evident if h is differentiable at X_t . \square

Step 3: Generalization to a local martingale \tilde{S} .

Proof of theorem 2.1– Let

$$H_t := \mathbb{E}[H_T | \mathcal{F}_t^X] .$$

Now, choose a localizing sequence $(\tau_n)_n$ such that the stopped processes X^{τ_n} and H^{τ_n} are in $L^2(\mathbb{P})$, for example by setting $\tau_n := \inf\{t : \|H_t\|_2^2 + \|X_t\|_2^2 \geq n\} \wedge \tau$. Note that

$$H_t^{\tau_n} = \mathbb{E}[H_T | \mathcal{F}_{\tau_n \wedge t}^X] = \mathbb{E}[H_T | \mathcal{F}_t^{X^{\tau_n}}] .$$

Following lemma 2.4, there exists for each n a process $\Delta^{(n)} = (\Delta^{(n),1}, \dots, \Delta^{(n),m}) \in L^2(\tilde{S}^{\tau_n})$ such that

$$H_T^n = H_0 + \sum_{i=1}^m \int_0^T \Delta_t^{(n),i} d\tilde{S}_t^{\tau_n, i} = H_0 + \sum_{i=1}^m \int_0^{\tau_n \wedge T} \Delta_t^i d\tilde{S}_t^i$$

where we have (consistently) defined $\Delta_t := \Delta_t^{(n)}$ whenever $\{\tau_n > t\}$. In other words, we have found a localizing sequence of stopping times $\tau_n \rightarrow T$ for Δ , which means that $\Delta \in L^{\text{loc}}(\tilde{S})$. \square

2.3 Complete Markets of Relevant Payoffs

We now turn to the financial application of the previous section, in particular theorem 2.1 on page 10 and its proposition 2.2 on page 11.

Our focus is now a diffusion $X = (S^1, \dots, S^n; A^1, \dots, A^r)$ of n tradable instruments and r state variables which is the unique solution to an SDE under the risk-neutral measure as

$$\begin{cases} dS_t^1 &= S_t^1 \mu_1(X_t) dt + S_t^1 \sum_{j=1}^d \sigma_1^j(X_t) dW_t^j \\ \vdots & \vdots \\ dS_t^n &= S_t^n \mu_n(X_t) dt + S_t^n \sum_{j=1}^d \sigma_n^j(X_t) dW_t^j \\ dA_t^1 &= \mu_{n+1}(X_t) dt \\ \vdots & \vdots \\ dA_t^r &= \mu_{n+r}(X_t) dt . \end{cases} \quad (17)$$

The fact that this is modelled under the risk-neutral measure means that the drift coefficients μ^1, \dots, μ^n for the tradable assets represent real proportional payments or costs arising from holding the i th asset.

The state variables A^1, \dots, A^r are used to carry additional information for the process: for example, choosing $A_0^1 := 0$ and the corresponding $\mu_{n+1} = 1$ implies that A_t^1 represents *time*: $A_t^1 = t$. Other choices for A are running averages or extrema, the quadratic variation of the returns of a stock price and other variables which are frequently used in finance.

Relevant Payoffs

As we have discussed before, we are now looking into assessing “completeness” of this model. Classic theorems essentially require that $d = n$ and that we can invert the volatility matrix uniformly (cf. theorem 2.2), which is especially difficult if the prices of tradable instruments can actually reach zero (CEV model, for example, or prices of credit default swaps). However, as we mentioned before, such a strong condition is only required if we really want to replicate *all* contingent claims which are measurable with respect to the rather artificial background driving Brownian motion W .

In reality, what we are interested in are those contingent claims whose value can be determined by observing the value of the traded instruments S^1, \dots, S^n – in other words of those contingent claims which are measurable with respect to the filtration generated by S or, equivalently, X .

DEFINITION 2.5 (Market of Relevant Payoffs) *We call the set $\mathbb{M}_r := \{H_T : H_T \in \mathcal{L}_+^1(\mathcal{F}_T)\}$ the market of relevant payoffs. A relevant payoff is a non-negative integrable random variable whose value depends on S , i.e. which is measurable with respect to some \mathcal{F}_T with $T < \tau$.*

This new concept is key since it means that we do no longer need to reproduce W in order to have access to a suitable representation theorem. Rather,

we can focus on a direct way on showing when the local martingale \tilde{S} is able to replicate all its relevant payoffs: following the discussion in section 2.1, we introduce the associated martingale processes

$$\tilde{S}_t^i := S_t^i - \int_0^t S_u^i \mu_i(X_u) dt .$$

With this setup we can directly apply theorem 2.1 and get:

COROLLARY 2.1 *Assume μ and σ in (17) are C^1 . Then, delta-hedging works, i.e. the market of all relevant payoffs is complete.*

Moreover, if the value function $h(t; s)$ for some H_T is $d\langle X^\tau \rangle \otimes d\mathbb{P}$ -almost surely differentiable, then

$$H_T = \mathbb{E}[H_T] + \sum_{i=1}^m \int_0^T \Delta_u^i d\tilde{S}_u^i$$

with $\Delta_t^i = \partial_{s^i} h(t; X_t)$.

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