Volatility and Dividends

Volatility Modelling with Cash Dividends and simple Credit Risk

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Abstract

This article shows how to incorporate cash dividends and credit risk into equity derivatives pricing and risk management. In essence, we show that in an arbitrage-free model the stock price process upon default *must* have the form

$$S_t = (F^*_t - D_t)X_t + D_t$$

where X is a (local) martingale with $X_0 = 1$, the curve F^* is the "risky" forward and D is the floor imposed on the stock price process in the form of appropriately discounted future dividends. This has already been shown in [1].

We show that the method presented is the only such method which is consistent with the assumption of cash dividends and simple credit risk. We discuss the implications for implied volatility, no-arbitrage conditions and we derive a version of Dupire's formula which handles cash dividend and credit risk properly.

We discuss pricing and risk management of European options, PDE methods and in quite some detail variance swaps and related derivatives such as gamma swaps, conditional variance swaps and corridor variance swaps. Indeed, to the our best if our knowledge, this is the first article which shows the correct handling of cash dividends when pricing variance swaps.

Keywords: Equity Volatility, Dividends, Credit Risk, Default, Implied Volatility, Local Volatility, Variance Swaps

Contents

1 General Setup

This article is about the very basics of equity modelling: the consistent incorporation of basic economic quantities into a simple, but efficient model of a share price process $S = (S_t)_{t\in\mathbb{R}_+}$ for the purpose of derivative pricing. We will focus on the following key ingredients into an asset's price evolution:

- The underlying interest rates.
- Borrowing ("repo") costs.
- Default risk.
- Dividends.
- The volatility structure.

We will take a pretty simplifying approach regarding most of these insofar as we assume that interest and borrowing rates, as well as default probabilities are deterministic and known in advance.

We will also use a particularly simple structure for the dividends of the stock: we assume that there is a series $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ of dividend dates, at each of which a proportional and then a fixed cash dividend is paid. We denote the proportional dividends by β_j and the cash dividends by α_j. That means that at any dividend date the holder of one share receives dividends worth a cash value of $S_{\tau_j-}\beta_j + \alpha_j$; we use S_{τ_j-} to refer to the stock price just before the payment of the dividend.[1] Obviously, a dividend is only paid if the stock does not default up to the dividend date.

In the absence of tax imbalances, no-arbitrage arguments show that at any ex dividend date τ_j our share price process $S = (S_t)_t$ with a value of S_{τ_j-} just before the dividend date must drop according to

$$S_{\tau_j} = S_{\tau_j-}(1 - \beta_j) - \alpha_j \ . \tag{1}$$

In practise, the jump will occur at the opening of trading at the ex-dividend day. The non-trading period between the close and the open can either be modelled using zero volatility or by scaling time accordingly, i.e. by modelling the asset in its business time. The former has the advantage to allow more easily for multi-asset structures which are traded in different time zones.

2 Cash Dividends in Equity Modelling

The first step is now to derive a general structure of equity share price models which are consistent with the setting we introduced above. Note that we are not intend to approximate a solution; rather, we derive the effects of introducing cash dividends for a share price process using purely no-arbitrage arguments.

[1] Note that we make the simplifying assumption that record date, ex-dividend date and payment date for the dividend all coincide.

The deterministic instantaneous interest rate is denoted by $r = (r_t)_t$. That means that the price at time t of a risk-free zero coupon bond with maturity T is given as

$$P(t,T) = e^{-\int_0^T r_s\,ds} .$$

Such a bond cannot default. In contrast, if the company issues a corporate zero bond (which will have zero recovery for the purpose of this article), an investor will demand a higher yield in compensation for the risk that the notional may actually not be paid.

Intuitively, in a market with diversifiable risk, we would expect the price of a risky bond to be equal to the price of a riskless bond times the (risk-neutral) probability of "survival" up to the maturity T of the instrument. Indeed, this is the case in our setup, which means that the value of a "risky" company bond provided it has not yet defaulted is given by

$$P^S(t,T) = P(t,T)\,\mathrm{SV}(t,T) ,$$

where $\mathrm{SV}(t,T)$ is the risk-neutral probability of survival up to T. At any point t, the function $T \mapsto \mathrm{SV}(t,T)$ is usually extracted from Credit Default Swap (CDS) market prices. We write the survival probability function often as

$$\mathrm{SV}(t,T) = e^{-\int_t^T h_s\,ds}$$

for some positive *credit spread* or *hazard rate* $h = (h_t)_t$. Since the default probabilities of the company for all maturities T are deterministic, h is also deterministic.

It is worthwhile noting that our inherent assumption by modelling default in this way is that we can not anticipate the default event itself by observing some publicly know variables (such as, say, the leverage of the company), but that the default is a sudden event (a fraud case, a sudden failure to honor a coupon payment on a bond etc). Mathematically, we say that the random default time τ is "inaccessible", which means it cannot not be the time at which some publicly observable variable such as the firm's debt hits a barrier.[2]

In general, it is far from obvious what exactly happens to a share price when such a default event occurs, but for sake of simplicity we will assume here that it drops to zero which makes our subsequent discussion particularly easy.

2.1 Consistent Share Price Processes

Assume that at some time $t > 0$, the stock has not yet defaulted and that we wish to enter into a forward contract with a maturity T. The standard way to compute the fair strike of the contract is by replication arguments: by re-investing all "relative" proceeds from holding the stock (repo and proportional dividends) to buy more shares, and by forward-selling the cash dividends we

[2]See Blanchet-Scalliet and Jeanblanc [2] for a detailed account on handling credit risk with intensities and Bermudez et al. [1] for the link with equity.

would receive from holding these shares, it is straightforward (and shown in appendix A.1) that the the fair strike of the forward contract must be

$$F(t,T) = \left(R(t,T)\, S_t - \sum_{j:t<\tau_j\le T} R(\tau_j,T)\ \alpha_j \right) \mathrm{SV}(t,T)\ , \tag{2}$$

where we used the "proportional growth factor"

$$R(t,T) := e^{\int_t^T (r_s+h_s-\mu_s)\,ds} \prod_{j:t<\tau_j\le T} (1-\beta_j)\ .$$

We will often abbreviate

$$F_t := F(0,t) \quad \text{and} \quad R_t := R(0,t)\ .$$

Note that the above formula (2) takes into account the possibility of a default after t and before T. One should note, in particular, that the forward strike differs from the standard Black&Scholes result in the presence of cash dividends and credit risk.

Given that a share bears no liability for the holder, the share price S_t can not become negative. This implies also that the forward can not be negative, because we would otherwise exchange at the maturity of the contract a negative cash amount for a share which will never have a negative value. That would clearly be an arbitrage situation. In other words, the forward is positive.

If we now apply this rather obvious observation to (2) and re-arrange for the stock price, then we obtain the condition

$$S_t \ge \sum_{j:t<\tau_j\le T} R(t,\tau_j)\ \alpha_j\ .$$

for $t > \tau$. Since this condition holds for *all* T, we have just shown:

PROPOSITION 2.1 *The value S_t of a stock can never fall below a floor given by the "growth rate-discounted" value of all remaining future cash dividends in the sense that*

$$S_t \ge D_t \quad \textit{where} \quad D_t := \sum_{j:\tau_j>t} \frac{\alpha_j}{R(t,\tau_j)} \qquad \{\tau > t\}\ . \tag{3}$$

This means that the share price process $S = (S_t)_t$ is floored by the deterministic process $D = (D_t)_t$.

This condition is also economically very sensible: the value of a share should always exceed the "growth rate-discounted" expected value of all forthcoming dividends, in particular if these are tradable via dividend swaps or calendar spreads of zero-strike calls.

While being economically sensible, it also imposes a new way of looking at the mathematical modeling of a share price process: given that the stock price at time t cannot be below $D_t > 0$, it does not make sense to model it as a random process which can get arbitrarily close to zero, such as the Black&Scholes model $S_t^{\mathrm{BS}} := F_t X_t^{\mathrm{BS}}$ where $X_t^{\mathrm{BS}} := \exp\left\{\sigma(t)W_t - \frac{1}{2}\sigma(t)^2 t\right\}$ with volatility σ and Brownian motion $(W_t)_{t\geq 0}$. Instead, the processes is much easier modeled consistently if a "stochastic model" such as X^{BS} is applied only to the remaining quantity at any time t which is exactly $F_t - D_t$ (without credit risk). *This* is the only portion of the stock price which is really random.

Assuming that X represents our general stochastic "volatility" model (i.e. local martingale), the key result of this article – and the related material in [1] – therefore really is the observation that the *only* consistently way to achieve this while making sure that the forward is priced correctly according to equation (2), is

$$S_t = \left\{ (F_t^* - D_t)\, X_t + D_t \right\} 1_{t<\tau} \ , \tag{4}$$

where we have use the "risk forward" $F_t^* := F_t/\mathrm{SV}(0,t)$, which is the fair strike of a forward which only settles if the company has not yet defaulted. This is explicitly stated in the following theorem.
The proof is presented on page 32 in appendix A.2.

THEOREM 2.1 (Stock price with Cash Dividends and Credit) *All arbitrage-free stock price processes for S can be written as*

$$S_t = 1_{\tau>t}\, S_t^* \tag{5}$$

with "non-defaulting stock price"

$$S_t^* := (F_t^* - D_t)\, X_t + D_t \tag{6}$$

where

- *The "pure stock price" $X = (X_t)_t$ is a non-negative (local) martingale with $X_0 = 1$.*[3]
- *$F_t^* := F_t/\mathrm{SV}(0,t)$ is the "risky forward" with $\mathbb{E}[F_t^* 1_{t<\tau}] = F_t$.*
- *The process $D = (D_t)_t$ represents the "growth-rate discounted" value of all future dividends from t onwards. It is a floor below S_t.*

NOTATION 1 (Summary of Notation) *The "risk forward" is given as*

$$F_t^* := \left(S_0 - \sum_{j:0<\tau_j\leq t} \alpha_j^* \right) R_t \ , \tag{7}$$

[3] Strictly speaking, the "all" above is only true if we assume "no free lunch with vanishing risk", (cf. Delbaen/Schachermayer [6]) which is stronger than "no arbitrage". For this text, we will also always assume that X is a true martingale.

This is the fair strike of a forward contract which settles only if the stock did not default until maturity. The formula is written in terms of the "growth rate-discounted dividends",

$$\alpha_j^* := \frac{\alpha_j}{R_{\tau_j}} \ .$$

Each α_j^* represent the time-zero value of all the cash dividends received per share bought at time 0 if all subsequent proportional proceeds from holding the share position are reinvested in the stock and if the stock did not default until the ex-dividend date τ_j.

Their value is consequently computed by "growth rate-discounting" the future cash dividend α_j using the proportional riskless growth factor of the asset,

$$R_t := e^{\int_0^t (r_s + h_s - \mu_s)\, ds} \prod_{j:\tau_j \le t} (1 - \beta_j) \ . \tag{8}$$

Following proposition 2.1, the value S_t of the share at some future time t cannot drop below the "discounted" value of all forthcoming future dividends, i.e. the stock price cannot fallow below the floor $D = (D_t)_t$ given as

$$D_t := R_t \sum_{j:\tau_j > t} \alpha_j^*$$

(cf. (3) on page 5).

Also note that (5) readily implies that

$$F_t = F_t^* \, \mathrm{SV}(0, t) \ .$$

Interpreation: The essence of the preceding result is that we can separate modelling the *volatility risk* X of the stock entirely from the other characteristics of the equity. In fact,

$$S_t^* = (F_t^* - D_t) X_t + D_t$$

really means that the only thing left to do is to specify the "pure" stock process X (hence the name). In this context, note that the process X itself should not carry any default risk (i.e., reach zero) because it would not actually imply default on the outstanding dividends: if $X_t = 0$, then the martingale property of X forces it to stay in 0, such that the stock S reduces to a pure risky cash bond paying the dividends as coupons.

Proposition 2.2 (Alternative Formulation) *An alternative representation for equation (6) in theorem 2.1 is the following: denote by*

$$S_0^* := S_0 - \sum_{j=1}^{\infty} \alpha_j^* \tag{9}$$

the "effective" stock price level which indicates the part of the initial stock price S_0 which is actually subject to volatility risk. Then, we obtain from (6) the formula

$$S_t^* := S_0^* R_t X_t + D_t \ , \tag{10}$$

which also implies $F_t^* = S_0^* R_t + D_t$.

Remark 2.1 (Business time) *To ease the implementation of business time such that the stock has zero volatility during non-trading hours, it is convenient to rescale time for X, i.e. write it in terms of a suitable martingale Z as*

$$X_t := Z_{A_t}$$

where $(A_t)_t$ is the business time process for the stock price. In particular, A does not increase outside trading hours. Moreover, it can also be used to model "event dates" where time is faster in the sense that the daily volatility is a multiple of a standard day (earnings announcements, index rebalancing, etc).

2.2 SDE for the Stock Price

In order to derive a simple form for the stochastic differential equation governing the dynamics of S, we start with the formulation of proposition 2.2: applying Ito's formula with jumps (cf. appendix B.1) to equation (10) yields[4] the following SDE for S^*:

$$\begin{aligned} dS^*_t &= S^*_{t-}(r_t + h_t - \mu_t)\,dt \\ &\quad + \left(S^*_{t-} - D_{t-}\right)\frac{dX_t}{X_{t-}} \\ &\quad - \sum\nolimits_j \left\{S^*_{t-}\beta_j + \alpha_j\right\}\delta_{\tau_j}(dt)\ . \end{aligned} \tag{11}$$

This equation separates the dynamics of the non-defaulting stock price S^* into its three main components: the first line represents the continuous growth of the stock, the second line its "randomness" (note that the randomness only applies to the excess of the stock level over the future dividends), and the last line represents the impact of dividend payments.

Accordingly, the SDE for S is then[5]

$$\begin{aligned} dS_t &= S_{t-}(r_t + h_t - \mu_t)\,dt \\ &\quad + (S_{t-} - 1_{\tau\geq t}\, D_{t-})\frac{dX_t}{X_{t-}} \\ &\quad - \sum\nolimits_j \{S_{t-}\beta_j + 1_{\tau\geq t}\,\alpha_j\}\,\delta_{\tau_j}(dt) \\ &\quad - S^*_{t-}\delta_\tau(dt). \end{aligned}$$

[4] The continuous differentials of R and D, respectively, are given as $dR^c_t = R_{t-}(r_t + h_t - \mu_t)\,dt$ and $dD^c_t = D_{t-}(r_t + h_t - \mu_t)\,dt$. Applying Ito to (10) hence gives

$$\begin{aligned} dS^*_t &= S^*_0 X_{t-} dR^c_t + S^*_0 R_{t-} X_{t-}\frac{dX_t}{X_{t-}} + dD^c_t + \sum\nolimits_j \left\{S^*_t - S^*_{t-}\right\}\delta_{\tau_j}(dt) \\ &= S^*_{t-}(r_t + h_t - \mu_t)\,dt + \left(S^*_{t-} - D_{t-}\right)\frac{dX_t}{X_{t-}} + \sum\nolimits_j \left\{S^*_{t-}(1-\beta_j) - \alpha_j - S^*_{t-}\right\}\delta_{\tau_j}(dt)\ , \end{aligned}$$

which yields the assertion after rearrangement (we implicitly assumed that $X > 0$ and that the jumps of X do a.s. not coincide with the dividend dates).

[5] From (5) we get $dS_t = 1_{\tau>t-}\, dS^*_t - S^*_{t-}\delta_\tau(dt)$.

Given our setup, the stock price is zero if and only if it defaulted, i.e. we can write the indicator $1_{\tau \geq t}$ equally as $1_{S_{t-}>0}$. This gives the following equation in terms of S_{t-} only and without reference to τ as

$$\begin{aligned} dS_t &= S_{t-}(r_t + h_t - \mu_t)\,dt \\ &\quad + \left(S_{t-} - 1_{S_{t-}>0}\,D_{t-}\right)\frac{dX_t}{X_{t-}} \\ &\quad - \sum\nolimits_j \left\{S_{t-}\beta_j + 1_{S_{t-}>0}\,\alpha_j\right\}\delta_{\tau_j}(dt) \\ &\quad - S_{t-}\delta_\tau(dt). \end{aligned} \tag{12}$$

Proposition 2.3 (Ito for continuous and positive X) *For the special case of a strictly positive, continuous pure diffusion*

$$\frac{dX_t}{X_t} = \sigma_{t-}\,dW_t$$

and a sufficiently smooth function H we can apply Ito (cf. appendix B.1) and get: [6]

$$\begin{aligned} dH_t(S_t) &= \Big\{H_t(0) - H_t(S_{t-})\Big\}\delta_\tau(dt) \\ &\quad + \partial_s H_t(S_{t-})\Big\{S_{t-}(r_t + h_t - \mu_t)\,dt + \left(S_{t-} - 1_{S_{t-}>0}\,D_{t-}\right)\sigma_{t-}\,dW_t\Big\} \\ &\quad + \frac{1}{2}\partial^2_{ss} H_t(S_{t-})\ \left(S_{t-} - 1_{S_{t-}>0}\,D_{t-}\right)^2\sigma^2_{t-}\,dt \\ &\quad + \sum\nolimits_j \Big\{H_t\big(S_{t-}(1-\beta_j) - \alpha_j 1_{S_{t-}>0}\big) - H_t(S_{t-})\Big\}\delta_{\tau_j}(dt) \\ &\quad + \partial_t H_t(S_{t-})\,dt \end{aligned} \tag{13}$$

$$\begin{aligned} &= \Big\{H_t(0) - H_t(S_{t-}) + S_{t-}\partial_s H_t(S_{t-})\Big\}\delta_\tau(dt) \\ &\quad + \partial_s H_t(S_{t-})dS_t \\ &\quad + \frac{1}{2}\partial^2_{ss} H_t(S_{t-})\ \left(S_{t-} - 1_{S_{t-}>0}\,D_{t-}\right)^2\sigma^2_{t-}\,dt \\ &\quad + \sum\nolimits_j \Big\{H_t\big(S_{t-}(1-\beta_j) - \alpha_j 1_{S_{t-}>0}\big) - H_t(S_{t-}) \\ &\qquad\qquad + \partial_s H_t(S_{t-})\,(S_{t-}\beta_j + \alpha_j)\Big\}\delta_{\tau_j}(dt) \\ &\quad + \partial_t H_t(S_{t-})\,dt \end{aligned} \tag{14}$$

Note that for $t > \tau$, these equations reduce to $dH_t(0) = \partial_t H_t(0)\,dt$.

2.3 The Total Return Process

The "total return" $S^{(\mathrm{TR})}$ of a stock S (most commonly an index) is generated by re-investing all dividends – but not any proceeds from repurchase agreements –

[6] We used the fact that the stock price is strictly positive at some time t if and only if default did not yet occur, i.e $1_{\tau \geq t} = 1_{S_{t-}>0}$.

back into the asset. That simply means that the value of the total return index will not drop at a dividend date. Therefore, the total return must have a form

$$S_t^{(\mathrm{TR})} = S_0 e^{\int_0^t (r_s + h_s - \mu_s)\, ds} Y_t \qquad \{\tau > t\} \tag{15}$$

where Y is a martingale. If Y is continuous, then the total return is essentially the "continuous part" of the stock price. Mathematically, this can be written as

$$S_t^{(\mathrm{TR})} = \int_0^t dS_{u-} \qquad \{\tau > t\} \ .$$

More generally, if t is between the dividend dates τ_k and τ_{k+1}, we have

$$S_t^{(\mathrm{TR})} = \frac{S_t}{S_{\tau_k}} \prod_{j=1}^{k} \frac{S_{\tau_k -}}{S_{\tau_{k-1}}} \qquad \{\tau > t\} \ .$$

We have therefore identified the martingale Y in equation (15) as

$$Y_t = \frac{S_t}{S_{\tau_k} R(\tau_k, t)}$$

whenever $\tau_k \le t < \tau_{k+1}$. In particular, Y is *not* equal to X.

Also note that $S^{(\mathrm{TR})}$ is not Markov with respect to itself: we need to simulate S in order to be able to compute a payoff depending on the total return.

REMARK 2.2 *Note that in the presence of cash dividends, the total return process is bounded from below away from zero. So it cannot be modeled using a Black&Scholes process or a similar standard process, either.*

3 European Options and Implied Volatility

Now that we have identified with equation (6) a formulation of the share price process which is consistent without our dividend assumptions, the question is which impact this has on the pricing and hedging of derivatives.

In this first section we will discuss the pricing of European options and various standard concepts such as implied and local volatility in the presence of credit risk and cash dividends. Section 4 is devoted mainly to PDEs and section 5 covers the pricing and risk management of variance swaps in the presence of credit risk and dividends.

Pure Options

We start with the observation that a European option on S is essentially a European option on X, since we can split each payoff H into its normal value and the payoff in the event of a default prior or at maturity:

$$H(S_T) = 1_{\tau > T}\, H\Big((F_T^* - D_T)\, X_T + D_T) \Big) + 1_{\tau \le T}\, H(0)$$

Now assume that we can observe market prices for call options on S for all strikes $K \geq 0$ and maturities $T \geq 0$, denoted by

$$C(T,K) := P(0,T)\,\mathbb{E}\left[\,\left(S_T - K\right)^+\,\right]\ .$$

We then define a call on the pure stock price X with "relative strike" x as

$$\mathbb{C}(T,x) := \mathbb{E}\left[\,(X_T - x)^+\,\right]\ .$$

Result 3.1 *The price of a call on the pure stock price X can be expressed in terms a call on S as*

$$\mathbb{C}(T,x) = \frac{1}{P^S(0,T)(F^*_T - D_T)}\,C\Big(T,(F^*_T - D_T)x + D_T\Big)\ . \tag{16}$$

Reversely,

$$C(T,K) = P^S(0,T)(F^*_T - D_T)\,\mathbb{C}\left(T,\frac{K - D_T}{F^*_T - D_T}\right)\ . \tag{17}$$

3.1 Dupire's Local Volatility with Cash Dividends and Credit Risk

In other words, we can obtain information on the pure stock price X by observing option prices on S: in particular, it means that the knowledge of all option prices for S for each maturity T gives us access to the marginal densities of the underlying pure martingale X via

$$\mathbb{P}[X_T = x] = \partial^2_{xx}\mathbb{C}(T,x)\ .$$

Accordingly, Dupire's implied local volatility [8] for the process X is given as

$$\varsigma^X_t(x)^2 := \frac{\partial_t\mathbb{C}(T,x)}{2\,x^2\partial^2_{xx}\mathbb{C}(T,x)}\ , \tag{18}$$

i.e. if the a unique solution to the SDE

$$\frac{dX_t}{X_t} = \varsigma^X_t(X_t)\,dW_t$$

exists, then it is a martingale and it reprices all European options in the sense that $\mathbb{E}[(X_t - x)^+] = \mathbb{C}(t,x)$ for all (t,x). The corresponding full stock price S will reprice the original option prices.

Using a local volatility for X rather than for S is much less prone to numerical instabilities due to dividend or credit effects, since all these effects have essentially been stripped out. For example, a local volatility function for S directly would by definition not be properly defined below the (time-dependent) lower bound D which would lead to numerical noise around this region. It is also often observed that the standard local volatility for S has severe numerical problems around dividend dates. Moreover, our "pure" formulation in conjunction with remark 2.1 allows the calibration of business-time local volatility.

Remark 3.1 *Assume that as above $dX_t/X_t = \varsigma_t(X_t)dW_t$. Then, we can rewrite the SDE (12) for the stock price as*

$$\begin{aligned} dS_t &= S_{t-}(r_t + h_t - \mu_t)\,dt + S_{t-}\sigma_t(S_{t-})\,dW_t \\ &\quad - \sum\nolimits_j \left\{ S_{t-}\beta_j + 1_{S_{t-}>0}\,\alpha_j \right\} \delta_{\tau_j}(dt) \\ &\quad - S_{t-}\delta_\tau(dt). \end{aligned} \tag{19}$$

with local volatility function

$$\sigma_t(s) := \begin{cases} \frac{s-D_{t-}}{s}\,\varsigma_t\left(\frac{s-D_{t-}}{F^*_{t-}-D_{t-}}\right) & (s > D_{t-}) \\ 0 & (s \le D_{t-}) \end{cases}. \tag{20}$$

(Note that as before the left limits on D and F^* indicate that these quantities shall not contain the dividends paid at t.)

3.2 Implications for Static No-Arbitrage Conditions

This is all well, but further investigation of equation (16) highlights a potential issue: we know that the imposition of cash dividends demands that the stock price S_T cannot fall below the floor D_T unless it defaults. Hence, there is a limit on the value of the call on the stock with a strike K of less than D_T: such a call must be equal to the discounted forward value, $C(T,K) = P(0,T)(F_T - K) = P^S(0,T)(F^*_T - K)$. This is a hard constraint on the value of calls with strikes below K which is not present when we use purely proportional dividends as in Black&Scholes.

In fact, it should come to no surprise that the introduction of cash dividends and credit risk changes the usual "no-arbitrage"-conditions. Consider, as another example, a call calendar spread over a dividend date: a call just after a dividend date is worth less than a call which is struck with the same strike just before the dividend. That means that the call prices are *not* always increasing in time.

The common no-arbitrage conditions such as positivity of butterflies and calendar spreads plus some constraints on the boundaries of course still hold, but not for S itself, but for the pure stock price X. They are summarized here:

Result 3.2 *Assume pure calls $\mathbb{C}(T,x)$ are given for all maturities $T \ge 0$ and all strikes $x \ge 0$. Then, X is a strictly positive true martingale if and only if*

1. *The forward is preserved, $\mathbb{C}(T,0) = 1$. This implies that $\mathbb{C}(T,0) = F_T$.*
2. *Butterflies have a positive value, $\partial^2_{xx}\mathbb{C}(T,x) \ge 0$.*
3. *Calendar spreads are positive, $\partial_T\mathbb{C}(T,x) \ge 0$.*
4. *The probability of reaching zero is nil, i.e. $\partial_x\mathbb{C}(T,0) = -1$.*[7]

[7] One could weaken this condition to $\partial_x\mathbb{C}(T,0) \ge -1$, in which case X can reach zero (for example, in CEV or SABR). However, as discussed above, we consider a stock price model in which the stock can degenerate into a coupon-bearing risky bond as not desirable.

5. *For high strikes we have* $\lim_{x\to\infty} \mathbb{C}(T,x) = 0$.

In this case we say that the market is "strictly free of arbitrage".[8]

The above summary is not only useful in the context of cash dividends; it also tells us that the slope of the call prices as a function of strike must change in the presence of credit risk. In particular, assume that there are no dividends but that there is a substantial credit risk, i.e. $\mathrm{SV}(0,T) \ll 1$. In this case, the original European option strip for S has the property

$$\frac{\partial_K C(T,0)}{P(0,T)} = -\mathrm{SV}(0,T) \gg -1 \ ,$$

hence the (undiscounted) call prices as a function of K must have a much higher slope than -1. Such a distribution cannot be generated by a diffusion. This means in particular that the standard measure of volatility risk, Black&Scholes' implied volatility, will explode as $K \downarrow 0$.[9] To recall, the *(standard) implied volatility of* S for strike K and maturity T is the positive number $\sigma_{\mathrm{BS}}(T,K)$ which solves

$$\mathbb{BS}\left(\sigma_{\mathrm{BS}}(T,K), T, \frac{K}{F_T}\right) \equiv \frac{C(T,K)}{P(0,T)F_T} \ . \tag{21}$$

where

$$\mathbb{BS}(\sigma,T,k) := \mathcal{N}(d_+) - k\mathcal{N}(d_-) \quad \text{with} \quad d_\pm = \frac{-\ln k \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \ .$$

Figure 1 shows the impact on Black&Scholes implied volatility if an otherwise flat 40% volatility is superimposed with either a 300 or a 1 bps credit spread. The former raises the 1m 90% implied volatility to 45%, and the ATM volatility with the same maturity to 43%, a full three volatility points higher than the base volatility of 40%. This shows that the effects of credit risk are not confined to far OTM puts, but can be seen already in prices for ATM options.

Moreover, the second graphs shows that the presence of even a minimal credit risk imposes very extreme lower bounds on implied volatility.

One might be tempted to argue that this observation does not really matter since options on the very short end are liquid, hence whether the implied volatility is purely the market's risk-adjusted volatility estimate or includes some credit risk should not matter. However, this argument ignores the effects the different effects have on hedging the position: if an option position on an underlying with credit risk is delta-hedged, the relevant volatility which determines the accuracy of this hedge will be the realized volatility of the pure stock price X (of 40% in the example above), for which the "mixed" standard implied volatility is not a valid indicator. The credit risk component of the option has to be hedged separately with CDS'.

[8] The case where only a discrete set of option prices is observed is equally straight-forward, cf. Buehler [5].

[9] See also Lee [10] where it is shown that the asymptotic growth rate of implied volatility is of at most of degree 2 in log-strike.

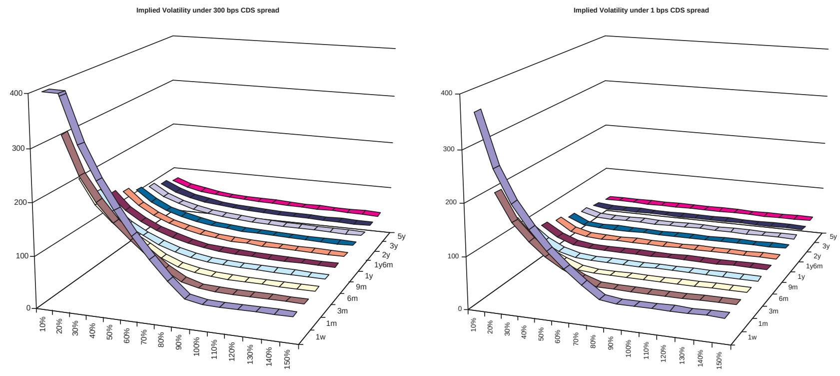

Figure 1: Black&Scholes implied volatilities for a hypothetical stock price with a CDS spread of 300bps (left) or only 1bps (right), superimposed on a flat pure stock price volatility of 40%. The credit spread of 300bps implies an annualized default probability of 4.9%, while the 1bps spread implies a risk of default per year of only 0.02%. It is remarkable that even such a very low probability of default has such a severe impact on short term implied volatilities.

3.3 Implied Volatility with Cash Dividends and Credit Risk

All in all, *standard* implied volatility in its simple form may not be the best way of seing the pure "volatility risk" in the stock price. Indeed, once we have accepted thinking of X as the volatile part of the stock price, it is much more natural to look at the implied volatilities of X directly. Thanks to equation (16) this is straightforward:

DEFINITION 3.1 *We call $\sigma_X(T,x)$ the "pure" implied volatility of S (or X) if it solves*

$$\mathbb{BS}\left(\sigma_X(T,x),T,x\right) \equiv \mathbb{C}(T,x) \ .$$

Note that this means that

$$C(T,K) = P^S(0,T)\left(F_T^* - D_T\right)\mathbb{BS}\left(\sigma_X\left(T,\frac{K-B_T}{F_T^*-D_T}\right),T,\frac{K-B_T}{F_T^*-D_T}\right) \ . \quad (22)$$

If our new measure of implied volatility is used, then it is possible to mark the implied volatility of X rather freely – for example, by using an implied volatility interpolation such as Gatheral's [9] or by specifying some martingale-dynamics such as Heston's for X – without violating the inherent arbitrage bounds on the standard implied volatility surface of S which are implied by the presence of credit risk and dividends. In particular, such a surface prevents trivial arbitrage between equity and credit instruments.

As an example, figure 2 compares the implied volatility surface of .FTSE100 with cash dividends for three years vs. purely proportional dividends.

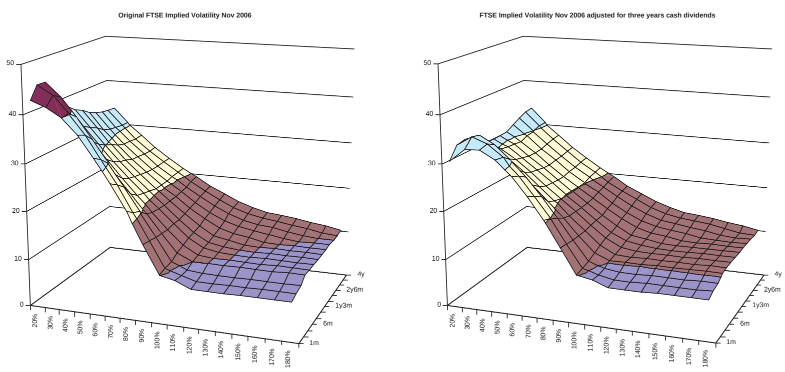

Figure 2: Original .FTSE100 implied volatility with proportional dividends (left) vs. the same data with cash dividends for three years (right). The forwards are identical in both cases. The right hand graph is produced by superimposing the cash dividends on the left hand volatility and then to to shift it in level to match the ATM curve.

Effects of Using Pure Volatility

The consequence of imposing cash dividends if the quoted option prices in the market are fixed is a higher pure implied volatility compared to the pure proportional case, in particular for lower strikes. This can be explained by considering an option with a strike K just above the floor D_T. While such an option will have a small but non-zero optionality value[10] for the purely proportional case for most volatilities, it will have virtually no optionality in the presence of cash dividends unless the pure volatility of X is very very high. Figure 3 shows the relation between "cash volatility" and "proportional volatility" in more detail.

The reverse conclusion is that if we mark the implied volatility of an underlying using its pure volatility, then the resulting standard implied volatility will actually be lower for low strikes, and it will become zero below the floor D_T.

While the introduction of cash dividends *lowers* the standard implied volatility if the same pure volatility is used, the opposite effect occurs, though, if we start to look at single stocks which also carry credit risk. In this case, the relationship between standard and pure implied volatility is reversed: even a small probability of default puts a non-zero probability point-mass at the default state zero, which lets standard implied volatilities explode.

4 PDE Pricing

In the previous section, we have discussed the implications of cash dividends and credit risk on the pricing of European options and consequences for local and implied volatility. In this section, we will expand the discussion to pricing

[10] excess of option value above intrinsic

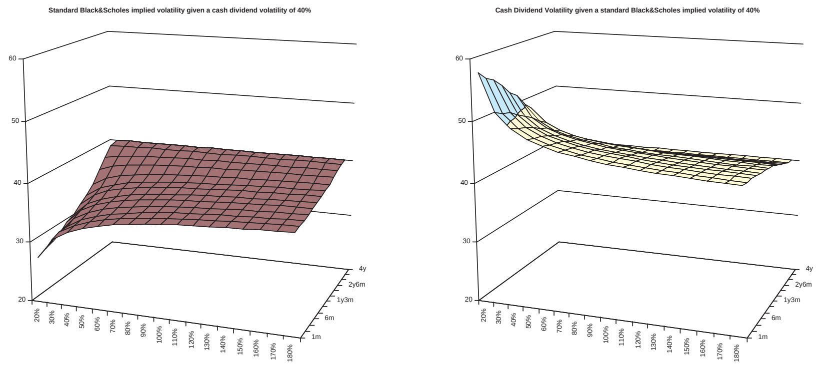

Figure 3: **Left:** standard Black&Scholes implied volatility surface if the stock price is modelled with cash dividends and a flat 40% volatility for X. The lower volatility on the short end is a reflection of the decreases variability of the stock price there. **Right:** the opposite transformation – the graph shows the implied volatility for X with cash dividends if the market is given as a flat 40% Black&Scholes volatility for all strikes and maturities. Accordingly, the effective volatility short end increases substantially.

with PDEs in our framework. We also briefly comment on Monte Carlo in section 4.4.

Setting

For simplicity of exposure, we assume that X is a local-volatility process, i.e.

$$\frac{dX_t}{X_t} = \varsigma_t(X_t)\, dW_t \tag{23}$$

as suggested in (18) above where ς is sufficiently well-behaved such that $X > 0$. We use the formulation suggested in remark 3.1 and define σ accordingly.

The following discussion can be extended easily to the case of higher-dimensional dynamics for the stock price.

4.1 The PDE of for the Full Stock Price Process

Let $H(S_T)$ be a payoff at maturity and let us assume first that there is no immediate recovery values for the option, i.e. the holder simply recevies $H(0)$ at maturity if the stock defaulted. Also note that the stock price S is zero if and only if it has defaulted, i.e. the stock price in the current setting is Markov. Set

$$H_t(s) := P(t,T)\, \mathbb{E}\left[\, H(S_T) \,\middle|\, S_t = s \,\right] \tag{24}$$

be the value of the claim at any time t. Using (13) we get

$$dH_t(S_t) \quad = \quad \Big\{ H_t(0) - H_t(S_{t-}) \Big\} \delta_\tau(dt)$$

$$
\begin{aligned}
&+\partial_s H_t(S_{t-})\Big\{S_{t-}(r_t+h_t-\mu_t)\,dt + S_{t-}\sigma_t(S_{t-})\,dW_t\Big\}\\
&+\frac{1}{2}\partial^2_{ss}H_t(S_{t-})\;S^2_{t-}\sigma_t(S_{t-})^2 dt\\
&+\sum\nolimits_j\Big\{H_t\big(S_{t-}(1-\beta_j)-\alpha_j 1_{S_{t-}>0}\big)-H_t(S_{t-})\Big\}\delta_{\tau_j}(dt)\\
&+\partial_t H_t(S_{t-})\,dt
\end{aligned}
$$

By construction, this must be a discounted martingale,[11] hence we can derive the associated PDE from the above SDE as

$$
\begin{aligned}
0 \;=\; & \partial_t H_t(s)\,dt - H_t(s)\,r_t + \partial_s H_t(s)\,s\,(r_t+h_t-\mu_t) + \frac{1}{2}\partial^2_{ss}H_t(s)\,s^2\sigma_t(s)^2\\
&+1_{s>0}\sum\nolimits_j\Big\{H_t\big(s(1-\beta_j)-\alpha_j\big)-H_t(s)\Big\}\delta_{\tau_j}(dt)\\
&+\big(H_t(0)-H_t(s)\big)h_t 1_{s>0}\;\;.
\end{aligned}
$$

This equation offers itself to straight forward interpretation of the dynamics of the stock price: the first line is simply the standard PDE for a local volatility-driven stock price with a modified drift which reflects the additional average growth rate which is to compensate the stock holder for the additional risk of default. The second line is the jump condition for the dividend payments.

The last line, finally, represents the infinitesimal probability of defaulting, in which case the value drops to $H_t(0)$. Note that the multiplication with the indicator $1_{s>0}$ is redundant in the case here, because the left hand bracket then equals zero anyway. However, the indicator becomes necessary if we extend the above formula to take into account "immediate recovery" values: this simply means that instead of falling to $H_t(0)$, the value of the security becomes, say, ρ_t. This quantity could technically also depend on the previous stock price level.

A good example of a security whose pricing requires such an additional recovery term ρ is a convertible bond which will default to the recovery value of its intrinsic bond, which is usually assumed to be around 40% of notional.

RESULT 4.1 *The PDE for an option with terminal payoff H and recovery process ρ is*

$$
\begin{aligned}
0 \;=\; & \partial_t H_t(s)\,dt - H_t(s)\,r_t + \partial_s H_t(s)\,s\,(r_t+h_t-\mu_t) + \frac{1}{2}\partial^2_{ss}H_t(s)\,s^2\sigma_t(s)^2\\
&+1_{s>0}\sum\nolimits_j\Big\{H_t\big(s(1-\beta_j)-\alpha_j\big)-H_t(s)\Big\}\delta_{\tau_j}(dt)\\
&+\big(\rho_t-H_t(s)\big)\,h_t\,1_{s>0}\;\;. \qquad (25)
\end{aligned}
$$

It should be noted that this equation also holds for the dynamics of the price process of the security after default: once default occurred, $s \equiv 0$, i.e. the value process of the option is trivial and has to accrue at the risk-free rate.

[11] To this end, note that $N_t := 1_{\tau>t} + \int_0^t h_s^{\wedge\tau}\,ds$ is a martingale: assume P is a Poisson-process with intensity h. Then, its compensated version $N'_t := P_t - \int_0^t h_s\,ds$ is a martingale. Now denote by τ' its first jump time, i.e. $\tau' := \inf\{t : N'_t = 1\}$ and define N'' as the stopped process N'. Then, N'' and $1-N$ have the same distribution.

4.2 American Options

American options are priced as usual: assume that upon exercise at time t, the option holder receives $E_t(S_{t-})$. the Accordingly, the respective partial differential inequality is given by

$$\begin{aligned}E_t(s) \quad \geq \quad & \partial_t H_t(s)\,dt - H_t(s)\,r_t + \partial_s H_t(s)\,s\,(r_t + h_t - \mu_t) + \frac{1}{2}\partial^2_{ss} H_t(s)\,s^2 \sigma_t(s)^2 \\ & + 1_{s>0} \sum_j \Big\{ H_t\big(s(1-\beta_j) - \alpha_j\big) - H_t(s) \Big\} \delta_{\tau_j}(dt) \\ & + \big(\rho_t - H_t(s)\big)\,h_t\,1_{s>0} \ .\end{aligned}$$

4.3 Pure Stock Price PDE

The previous PDE (25) is the natural description of the asset dynamics, but when implemented numerically, it has two major drawbacks: first is the need to implement a jump condition on the finite difference grid at every dividend date which will slow down convergence – in particular, when used with indices which may pay dividends nearly every day such as S&P500. Secondly, (25) requires at least in its pure application the evaluation of unnatural "instantaneous" quantities such as r_t and h_t. Unless we actually require the value of the option at any time between now and maturity (continuous exercise conditions or continuous Barriers), it is therefore often more efficient to express the price of an option on S as a price of an option on X.

To this end, consider a Bermudan option that can be exercised on dates $0 = T_0 < \cdots < T_N$ with exercise values $E_0(s), \ldots, E_N(s)$ depending on the stock price s at that time. We assume that the exercise decision is communicated after any dividends are paid on the stock. In the event of default, the holder of the option recovers ρ_t.

At maturity, assuming no default, the value of the option is obviously

$$H_{T_N}(s) = \max\{E_N(s),\, 0\} \ .$$

At any exercise date T_ℓ before maturity and default, standard pricing theory tells us that the value of the option will be the maximum between exercising the option now or holding on to it. If we hold on to the option, the stock may default before the next exercise decision. Hence, the holding value must be the sum of the discounted value of the next exercise date, multiplied by the survival probability, plus the value of the recovery in the event of default, weighted by the default probability. We write this as

$$H_{T_\ell}(s) := \max\Big\{ E_\ell(s),\, C_\ell(s) + G_\ell \Big\} \ ,$$

where

$$C_\ell(s) := P^S(T_\ell, T_{\ell+1})\,\mathbb{E}\left[H_{T_{\ell+1}}(S_{T_{\ell+1}}) \,\middle|\, S_{T_\ell} = s \right] \tag{26}$$

denotes the value of holding the option if it does not default until the next exercise time, and where

$$G_\ell := \mathbb{E}\left[\, P(T_\ell,\tau)\rho_\tau 1_{T_\ell<\tau\le T_{\ell+1}} \,\middle|\, \tau > T_\ell \,\right] = \int_{T_\ell}^{T_{\ell+1}} h_t\rho_t P^S(T_\ell,t)\, dt$$

denotes the recovery value if it does. Note that G_ℓ is deterministic in our current framework. To compute (26), define for $t \in [T_\ell, T_{\ell+1}]$

$$\tilde{C}_t(x) := \mathbb{E}\left[\, H_{T_{\ell+1}}\Big((F^*_{T_{\ell+1}} - D_{T_{\ell+1}})X_{T_{\ell+1}} + D_{T_{\ell+1}}\Big) \,\Big|\, X_t = x \,\right] \tag{27}$$

such that

$$C_\ell(s) = P^S(T_\ell, T_{\ell+1})\ \tilde{C}_{T_\ell}\left(\frac{s - D_{T_\ell}}{F^*_{T_\ell} - D_{T_\ell}}\right) \ .$$

Under the local volatility assumption (23) for X, the PDE for (27) is as usual given by

$$0 = \partial_t \tilde{C}_t(x)\, dt + \frac{1}{2}\partial^2_{xx}\tilde{C}_t(x)\, x^2 \varsigma_t(x)^2 \ ,$$

which is much easier to implement efficiently, in particular if ς is given analytically via Dupire's formula (18).

4.4 Monte-Carlo

From an implementation point of view, Monte-Carlo methods are often more intuitive even if they suffer from less efficiency for low dimensions. However, for dimensions above two or for strongly path-dependent products, there is virtually no alternative so far for the evaluation of exotic options.

In our framework, the implementation of an efficient Monte-Carlo scheme can be achieved by separating the simulation of X which depends on the choice of the stochastic dynamics and the subsequent application of dividend and credit effects. In conjunction with remark 2.1 on business time, such an implementation can be transparent from a modeling point of view: the handling of X does not require any additional information on S. For example, the transformations described in result 3.1 can be used to provide a "pure" implied volatility surface for the calibration of the stochastic process X.

5 Variance Swaps

While the previous concepts are straight-forward for simple European options – after all, we simply apply an affine transformation to the stock price –, the relationship between S and X of course also impacts the pricing and risk management of variance swaps.

This is the topic of the current section. We will start by introducing the concepts of variance swaps, discuss the impact of cash dividends and then additionally how to take into account default risk.

Definitions

The basic idea of a variance swap is to pay the annualized realized variance, i.e. the squared volatility of the returns of the underlying asset in exchange for a previously agreed fixed "set volatility", also squared. The definition of realized variance over close-of business days $0 = t_0 < \cdots < t_n = T$ is typically simply the standard mathematical definition,

$$\mathrm{RV}^I(T) := \sum_{i=1}^n \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 .$$

However, to avoid that dividends impact the computation of realized variancre, single-stock variance swaps are usually based on a dividend-adjusted version,

$$\mathrm{RV}^S(T) := \sum_{i=1}^n \left(\log \frac{S^{(\mathrm{TR})}_{t_i}}{S^{(\mathrm{TR})}_{t_{i-1}}} \right)^2 ,$$

which is just a clever way of saying that the returns of S as adjusted for the dividends.[12]

With either definition of realized variance, the payoff of a variance swap is

$$\frac{252}{n}\mathrm{RV}(T) - K^2$$

where K represents the fixed leg's "set volatility". The factor $252/n$ annualizes the variance assuming that the year has 252 business days. The number 252 is usually contractually fixed.

Remark 5.1 *The payoff of single-stock variance swaps is usually also subject to an additional cap at* 150% *of* K^2 *since very extreme stock price movements, for example due to takeover rumors or default, would otherwise have a very strong effect on the realized variance estimator. This turns the contract into a non-linear payoff on realized variance.*

For the time being, we will ignore the cap; cf. Buehler [3] on how to model the cap value.

For pricing purposes, it should be clear that the fixed leg of a variance swap is a trivial zero bond. Moreover, the scaling factor $252/n$ in front of realized variance simply changes the notional. Hence, when talking about "variance swaps", we will for most part assume that K is zero (a so-called "zero-strike variance swap") and that the scaling factor is 1. In other words, a variance swap henceforth simply pays realized variance to the holder unless otherwise stated.

[12] Assuming that dividend effects occur end-of day, $\mathrm{RV}^S(T) = \sum_{i=1}^n \left(\log S_{t_i-}/S_{t_{i-1}} \right)^2$.

5.1 Classic Variance Swap Pricing

The classic approach to price a (not annualized zero-strike) variance swap is to assume that X has no jumps, that S pays no cash dividends, that the repo-rate is zero and that there is no risk of default. Then, Ito's formula applied to the logarithm of the stock for each interval $[t_{i-1}, t_i)$ yields

$$\begin{aligned} \log \frac{S_{t_i}}{S_{t_{i-1}}} &= \int_{t_{i-1}}^{t_i} \frac{dS_t}{S_t} - \frac{1}{2} \int_{t_{i-1}}^{t_i} \frac{d\langle S \rangle_t}{S_t^2} \\ &\approx \frac{1}{S_{t_{i-1}}} \left(S_{t_i} - S_{t_{i-1}} \right) - \frac{1}{2} \frac{1}{S_{t_{i-1}}^2} \left(S_{t_i} - S_{t_{i-1}} \right)^2 \ . \end{aligned} \tag{28}$$

Summing these terms up and rearranging them gives Neuberger's [11] famous formula

$$\mathrm{RV}^I(T) \approx -2 \log \frac{S_T}{S_0} + 2 \sum_{i=1}^{n} \frac{1}{S_{ti-1}} \left(S_{t_i} - S_{t_{i-1}} \right) \ . \tag{29}$$

This formula suggests the following approach: to replicate the left hand side, short two European log-contracts (we comment on this below) and execute a daily delta-hedge which results in the last term in (29), following the remarks in Bermudez et al. [1]. This means running a delta of

$$\Delta_i^{\text{hedge}} := 2 \frac{P(t_i, T)}{S_{t_{i-1}}} \ , \tag{30}$$

which amounts to a "cash delta" of 2, properly discounted ("cash delta" denotes the amount of capital invested in the stock, i.e. it is the standard delta times the stock price). Note, however, that this strategy is not self-financing if rates are not zero. Its costs are covered by the discounted value of the last term in equation (29).[13]

5.1.1 Intra-day Pricing and Risk Management

A theoretically slightly more accurate approach than using (29) is to use a continuous form of that expression. The basic idea is that RV^I is an unbiased estimator for the quadratic variation of the log-returns of the stock, $\mathrm{QV}(t) := \langle \log S \rangle_t$ even if S has jumps. This means that

$$\mathbb{E}\left[\mathrm{RV}^I(T) \,\middle|\, \mathcal{F}_{t_i} \right] = \mathbb{E}\left[\mathrm{QV}(t_i, T) \,\middle|\, \mathcal{F}_{t_i} \right] + \mathrm{RV}^I(t_i) \tag{31}$$

for all $i = 0, \ldots, N$, where we used $\mathrm{QV}(T_1, T_2) := \mathrm{QV}(T_2) - \mathrm{QV}(T_1)$.

[13] At the beginning of each period, we borrow a cash delta amount of $2P(t_i, T)$ and invest it into the stock for a price of $S_{t_{i-1}}$. At the end of the period, we sell the stock position for S_{t_i}, hence our P&L is $2P(t_i, T) S_{t_i}/S_{t_{i-1}} - 2P(t_i, T)/P(t_{i-1}, t_i)$, which translates at maturity into

$$2\left(S_{t_i}/S_{t_{i-1}} - 1 \right) - 2\left(1/P(t_{i-1}, t_i) - 1 \right) = 2\left(S_{t_i}/S_{t_{i-1}} - 1 \right) - 2\left(F_{t_i}/F_{t_{i-1}} - 1 \right) \ ,$$

where the latter term matches exactly the expectation of the last term in (29).

Now, we simply apply Ito to the log-contract and rearrange accordingly. This gives:

$$\mathrm{QV}(t,T) = -2\log\frac{S_T}{S_t} + 2\int_t^T \frac{dS_u}{S_u} \ , \tag{32}$$

which for the case $t=0$ essentially coincides under expectation with (29). Note that in the absence of credit risk and cash dividends, (11) shows that

$$\mathrm{QV}(t,T) = -2\log\frac{S_T}{F(t,T)} + 2\int_t^T \frac{dX_u}{X_u} \ . \tag{33}$$

To finally evaluate a variance swap at a point in time $t \in [t_{i-1}, t_i]$, we split the computation of the expected payout into three distinct parts:

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{RV}^I(T)\,\middle|\,\mathcal{F}_t\right] \\
&= \sum_{k=1}^{i-1}\left(\log\frac{S_{t_k}}{S_{t_{k-1}}}\right)^2 + \mathbb{E}\left[\left(\log\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2\,\middle|\,\mathcal{F}_t\right] + \mathbb{E}\left[\sum_{k=i+1}^{n}\left(\log\frac{S_{t_k}}{S_{t_{k-1}}}\right)^2\,\middle|\,\mathcal{F}_t\right] \\
&= \underbrace{\mathrm{RV}^I(t_{i-1})}_{\text{Past}} + \underbrace{\mathbb{E}\left[\left(\log\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2\,\middle|\,\mathcal{F}_t\right]}_{\text{Present}} + \underbrace{\mathbb{E}\left[\mathrm{QV}(t_i,T)\,\middle|\,\mathcal{F}_t\right]}_{\text{Future}}
\end{aligned} \tag{34}$$

While the "Past" is trivial to compute in theory (apart from practical problems such as stock splits, non-trading days and other real-life phenomena), the question remains how to compute "Present" and "Future". Since the main idea of splitting the payoff is to guarantee that the pricing scheme returns the correct cash delta, it is in fact sufficient to compute the "Present" with a zero-volatility Black&Scholes model, in which case the "Future" is in the light of (33) simply given by

$$P(t,T)\,\mathbb{E}\left[\mathrm{QV}(t_i,T)\,\middle|\,\mathcal{F}_t\right] = -2\,P(t,T)\,\mathbb{E}\left[\log\frac{S_T}{F(t,T)}\,\middle|\,\mathcal{F}_t\right] \ .$$

This formula is also true if purely proportional dividends are present. It does not hold in the presence of cash dividends or credit risk, on which we comment below.

5.1.2 Realized Variance vs. Quadratic Variation

The approximation of realized variance by quadratic variation is valid only for linear payoffs on variance. The price of any non-linear payoff such as a call on realized variance must be computed based on realized variance. To see this, assume that X is a Black&Scholes process and that $S = X$. Following Christopher Jordinson's argument presented on page 12ff. in [4] we then have

$$\frac{\mathrm{RV}(0,T)}{\mathrm{QV}(0,T)} = \sum_{i=1}^{n}\left\{\frac{x_i - \frac{1}{2}\sigma\sqrt{T/n}}{\sqrt{n}}\right\}^2$$

with $(x_i)_{i=1,\ldots,n}$ iid standard normal. Consequently, the ratio RV/QV is chi-square distributed with parameter $\lambda = \sigma^2 T/4n^2$.

Figure 4 illustrates the effect on option prices in a Heston model: on the short end the value of an option written on quadratic variation declines while it increases for options on realized variance.

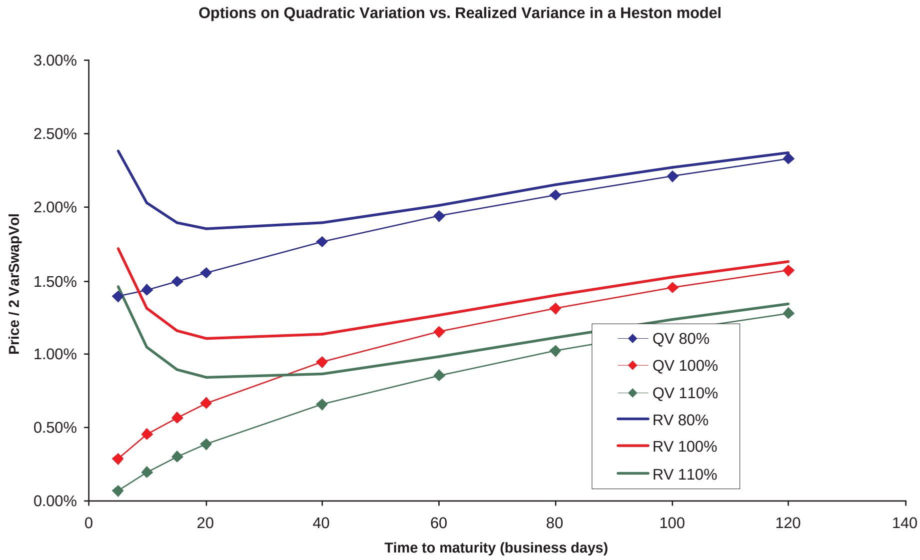

Figure 4: Options on realized variance vs. options on quadratic variation in a Heston model. The graph illustrates that short term options on realized variance cannot be priced using quadratic variation.

5.1.3 The Delta of a Variance Swap in Black&Scholes is Zero

From (34), we can easily derive the delta of a variance swap as

$$\Delta_t^{\mathrm{VS}} = 2\frac{P(t,T)}{S_t}\log\frac{S_t}{S_{t_{i-1}}} + \partial_{S_t}P(t,T)\underbrace{\mathbb{E}\left[\log\frac{S_T}{F(t,T)}\,\middle|\,\mathcal{F}_t\right]}_{(*)}. \qquad (35)$$

Here, it is important to notice that the term $\log S_T/F(t,T)$ does not depend on the current stock price level, S_t. That means any sensitivity of $(*)$ towards S_t must be via a dependency of the volatility surface on the stock price level, as it is the case for local volatility models. However, for all 'classic stochastic volatility models, jump models or, in fact, Black&Scholes, where the dynamics of the underlying volatility structure do not functionally depend on S_t (or X_t for that matter), the term $(*)$ will have no delta.

REMARK 5.2 *The delta of a variance swap in Black&Scholes at any fixing date* $t = t_i$ *is zero,*

$$\Delta_{t_i}^{\mathrm{VS}} = 0 \ .$$

The same is true for all classic stochastic volatility and jump models.

The famous delta-term (30) which we quoted above is the result of offsetting a variance swap position with the respective log-contract: assume that we add a log-contract initiated at time $t = 0$ to the payoff (34). This contract is

$$\mathbb{E}\left[\log \frac{S_T}{F(0,T)} \,|\, \mathcal{F}_t\right] = \mathbb{E}\left[\log \frac{S_T}{F(t,T)} \,|\, \mathcal{F}_t\right] + \log \frac{S_t}{S_0} - \log P(0,t)$$

where we used the fact that $F(t,T)/F(0,T) = S_t/S_0/P(0,t)$ in our current setting. If we add the delta of this term to (35) we end up at any fixing date t_i with exactly the previously mentioned

$$\Delta_i^{\mathrm{hedge}} = 2\frac{P(t_i,T)}{S_{t_i}} \ .$$

From a booking perspective, this implies that the variance swap and its log-contract hedge (or any approximation to it) should be booked into the same account in order to make sure that the system reports the right combined stock sensitivity. Note that such a procedure also allows to book much reduced "log contracts" against the variance swap, such as simple butterflies, while the price of the variance swap is still computed using the full log-contract.

REMARK 5.3 *If the log-contract* $(*)$ *is computed using the approximation approach discussed below, then a typical risk-management system will produce a delta for this term based on the rules governing the change of implied volatility if the stock price changes. In this sense, the delta of a variance swap in such a system is a pure "dVol/dSpot" measure.*

Log-contracts

The above discussion and in particular the replication formula (29) assumes that we can actually trade log-contracts. In practise, this is not the case. If we were able to trade arbitrary amounts of European options with all strikes, we could replicate any European payoff by virtue of the following formula:

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \int_{x_0}^{\infty} f''(z)(x-z)^+ \, dz + \int_0^{x_0} f''(z)(z-x)^+ \, dy \, dz \ . \tag{36}$$

For the log-contract, we obtain:

$$\log \frac{S_T}{F_T} \quad = \quad \frac{S_T - F_T}{F_T} + \int_{F_T}^{\infty} \frac{(S_T - K)^+}{K^2} \, dK + \int_0^{F_T} \frac{(K - S_T)^+}{K^2} \, dK \ .$$

It is clearly not convenient to trade in an infinite number of contracts, hence the above expression needs to be approximated by using finitely many European options. Assume we are able to trade in options with strikes $0 < K_{-n_p} < \cdots < K_0 < \cdots < K_{n_c}$ where $K_0 = F_T$ is ATM. Since $\log x$ is concave, we can construct a following sub-hedge on the interval $[K_{-n_p}, K_{n_c}]$:

$$\log S_T \geq \log F_T + \sum_{k=1}^{n^c} w_k^c \, (S_T - K_{k-1})^+ + \sum_{k=1}^{n^p} w_k^p \left(K_{-(k-1)} - S_T\right)^+ \ ,$$

with appropriate weights w^c and w^p, cf. appendix B.2. A super-hedge can be constructed in a similar way.

A common point of concern is the fact that this super-hedge needs to be cut off at some point. The above formula is very sensitive to price of OTM options, which in turn are usually very illiquid. Hence, the price of a variance swap can not usually be read from the price of traded options, even under the idealizing assumptions of deterministic rates, no credit risk and no cash dividends.

In practise, of course, there are many more effects which are not taken into account by such a simplifying framework, chiefly among them riskiness of interest rates.

5.1.4 Interpretation: Playing Implied vs. Realized Volatility

Consider the following idea: let us assume that we want to hedge an European option with payoff $H \geq 0$ using the Black&Scholes model: we fix some "set volatility" K and compute, at any time t, the price of the option as $\Pi_t^{\mathrm{BS}}(S_t)$ using the volatility K. We then short the option and delta-hedge according to Black&Scholes's delta $\Delta_t^{\mathrm{BS}}(S) := \partial_S \Pi_t^{\mathrm{BS}}(S)$, again computed with our "set volatility" K.

Assuming the real stock price is a diffusion

$$\frac{dS_t}{S_t} = (r_t - \mu_t)\, dt + \sigma_t \, dW_t$$

with true (stochastic) volatility σ, Ito shows that

$$d\Pi_t^{\mathrm{BS}}(S_t) = \Delta\Pi_t^{\mathrm{BS}} \, dS_t + \frac{1}{2}\Gamma_t^{\mathrm{BS}} \, \sigma_t^2 S_t^2 \, dt + \theta_t^{\mathrm{BS}} \, dt$$

where we used Gamma $\Gamma_t^{\mathrm{BS}}(S) := \partial_{SS}^2 \Pi_t^{\mathrm{BS}}(S)$ and Theta $\theta_t^{\mathrm{BS}}(S) := \partial_t \Pi_t^{\mathrm{BS}}(S)$. Since Π satisfies the Black&Scholes-PDE, we have that

$$\theta_t^{\mathrm{BS}}(S) = -\frac{1}{2}\Gamma_t^{\mathrm{BS}} \, K^2 S^2 \ ,$$

hence we get the standard result that our accumulative hedging error is

$$\begin{aligned} \mathrm{P\&L}_{\mathrm{BS}}(H) &:= -\Big(H(S_T) - \Pi_0(S_0)\Big) + \int_0^T \Delta_t^{\mathrm{BS}} \, dS_t \\ &= \frac{1}{2} \int_0^T \Gamma_t^{\mathrm{BS}} S_t^2 \left(\sigma_t^2 - K^2\right) \, dt \ . \end{aligned}$$

That means that under the assumption that the real stock price is a diffusion, we can bet on a discrepancy between implied squared volatility K^2 and true squared volatility σ^2: if we assume, for example, that the quoted implied volatility of some call is probably going to be below the true volatility of the asset, we could buy that call and delta-hedge it with its initial implied volatility. Since the call's gamma is positive, $\mathrm{P\&L_{BS}}$ will be positive if the true volatility is indeed always higher than the implied volatility.

This approach, however, has one drawback: Cash Gamma $\Gamma_t^{\mathrm{BS},\$} := \Gamma_t^{\mathrm{BS}} S_t^2$ of a call is not a constant. Hence, the gain we make if we are right in our prediction will depend a lot on *when* the true volatility is above the implied volatility. This is particularly a concern if the spot moves away from the strike of the call in which case gamma decays rapidly and the effect of our bet diminishes. It also means that even if we are on average right that the true volatility is above the implied volatility, we may well still make a loss with the above strategy.

To alleviate this drawback, we would therefore require an option whose Cash Gamma is independent of the spot level, for example a constant. This payoff now is exactly the log-contract, $H(S_T) := -2\log S_T$, i.e.

$$\mathrm{P\&L_{BS}}(2\log) \quad = \quad \int_0^T \left(\sigma_t^2 - K^2\right)\, dt \; .$$

Moreover, the delta of a log-contract *in Black-Scholes* is easily seen to be exactly $\Delta_t^{\mathrm{BS}} = P(t,T)/S_t$ regardless of the implied volatility. This is in line with our formula (32).[14] In this sense, a variance swap can be seen as the delta-hedged European option position which provides a constant exposure in cash to the discrepancy between implied and true volatility.

Remark 5.4 (Delta of a Variance Swap in Black-Scholes) *In Black-Scholes, a variance swap which pays pure quadratic variation has* no delta *– its payoff is a constant. However, the delta of a short position in such a variance swap, plus a long position in its log-hedge and the appropriate funding in cash (see below) will be* $2P(T,t)/S_t$.

However, even in Black-Scholes a standard variance swap paying realized variance will have a residual daily delta which arises from the daily log-squared returns which comprise its payoff, as can be seen in (34).

Remark 5.5 (Delta of a Variance Swap in Stochastic Volatility models) *In other volatility models, such as Dupire's implied local volatility, a variance swap will have some delta arising from the additional dependence of volatility on spot. In stochastic volatility models such as Heston's, a variance swap has only a very small delta which arises from the fact that it pays the realized variance rather than quadratic variation. If it were to pay just the latter, it wouldn't have a delta in the classical sense, but a "vega" towards a volatility which is highly correlated with spot.*

[14] More details on this construction can be found in Demeterfi et al. [7].

5.2 Cash Dividends

The presence of cash dividends alters the argumentation above only slightly. The main difference is that then (28) on page 21 does not hold since it omits the jumps due to the dividends. Assuming that X itself has no jumps, we obtain

$$\begin{aligned} \log \frac{S_{t_i}}{S_{t_{i-1}}} &= \int_{t_{i-1}}^{t_i} \frac{dS_t}{S_t} - \frac{1}{2} \int_{t_{i-1}}^{t_i} \frac{d\langle S \rangle_t}{S_t^2} \\ &+ \log \frac{S_{t_i}}{S_{t_i-}} - \left(\frac{S_{t_i}}{S_{t_i-}} - 1 \right) - \left(\frac{S_{t_i}}{S_{t_i-}} - 1 \right)^2 , \end{aligned}$$

which has also been discussed in [1] and Buehler [4]. Following (1) on page 3 we can write at each dividend date

$$S_{\tau_j -} = \frac{S_{\tau_j} + \alpha_j}{1 - \beta_j} . \tag{37}$$

(we here use the assumption that dividends are paid right before the open). Let us also introduce the function

$$D_j(s) := \log \frac{s\,(1-\beta_j)}{s+\alpha_j} + \frac{s\,\beta_j + \alpha_j}{s+\alpha_j} - \left(\frac{s\,\beta_j + \alpha_j}{s+\alpha_j} \right)^2 .$$

Then, we can summarize just as above that the payoff of a variance swap which is not adjusted for dividends is given as

$$\mathrm{RV}^I(T) = -2 \log \frac{S_T}{S_0} + 2 \sum_{i=1}^n \left(\frac{S_{t_i}}{S_{t_{i-1}}} - 1 \right) + 2 \sum_{j:\tau_j \le T} D_j(S_{\tau_j}) . \tag{38}$$

The key in this expression is that the additional term on the right side is simply a strip of European payoffs, which in turn can again be approximated by (36). If dividends are taken out of the computation of the variance swap returns, the formula changes accordingly to

$$\mathrm{RV}^S(T) = -2 \log \frac{S_T}{S_0} + 2 \sum_{i=1}^n \left(\frac{S_{t_i}}{S_{t_{i-1}}} - 1 \right) + 2 \sum_{j:\tau_j \le T} D_j^S(S_{t_i}) ,$$

where

$$D_j^S(s) := \log \frac{s\,(1-\beta_j)}{s+\alpha_j} + \frac{s\,\beta_j + \alpha_j}{s+\alpha_j} ,$$

i.e. we omit the square term from D.

Remark 5.6 (**Are variance swaps cheaper with cash dividends?**) *A very common question will be to ask whether variance swaps become more or less expensive if cash dividends are introduced.*

It needs to be stressed that one can not *simply price a variance with and without cash dividends off the same implied volatility surface: in the light of our* *discussion above it should be clear that changing the dividend assumption also changes the implied vol surface – or, in other words, if the implied vol surface is marked with one dividend assumption, then we should not simply price a variance swap with another assumption.*

By experience[15]*, however, prices of variance swaps fall if cash dividends are introduced if the vol-surfaces are re-fitted consistently with observed option prices.*

5.3 Credit Risk

The proper incorporation of credit risk into variance swap pricing in our framework is actually quite straight forward: by definition of the contract, realized variance will become infinite, hence the contract's payoff will simply be the maximum capped amount, $150\%K^2$, cf. remark 5.1 on page 20.

In other words, following a suggestion in Buehler [4], we can write the single stock variance swap payoff as

$$\mathit{1}_{\tau>T}\left(\frac{252}{n}\mathrm{RV}^S(T)-K^2\right)+\mathit{1}_{\tau<T}\,150\%K^2\ ,$$

which can be achieved easily using the methods described above.

However, in practise one should keep in mind the typically very high correlation between volatility and default intensities. Hence, if modelling variance swaps in single stocks, it is advisable to incorporate stochastic default rates. The above will also not work if variance spikes after the inception of the product. In this case, the cap will have true optionality and needs to be evaluated with elaborate methods, such as those discussed in [1] and in more detail in [3].

5.4 Related Products

The pricing of variance swaps can be extended to a series of products with very similar characteristics, for example "Gamma swaps" and "Conditional Variance Swaps" or "Corridor Variance Swaps". The idea in all cases is the same, as the payoff is now

$$\frac{1}{n}\sum_{i=1}^{n}\left\{252\ f(S_{t_{i-1}})\left(\log\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2-g(S_{t_{i-1}})K^2\right\}\ . \tag{39}$$

For example, a variance swap has $f(s)=1$ and $g(s)=1$, while a *gamma swap* has $f(s):=s$ and $g(s)=1$. A *conditional variance swap* with a lower barrier of B has $f(s)=\mathit{1}_{s>B}$ and $g(s)=\mathit{1}_{s>B}/n$, i.e. it is a variance swap which only pays out realized variance vs. set variance for those days where the stock price trades above the barrier. A *corridor variance swap*, finally, is given by $f(s)=\mathit{1}_{s>B}$ and $g(s)=1$, i.e the fixed leg is always paid while the floating

[15] prior to 2008

leg is only paid if the stock price trades above the barrier B. Cf. [1] for some more information.

From a risk-management point of view, this product again separates into two: the right hand side is a strip of European options, mostly digitals, for which most desks have a well-developed pricing methodology.

The left hand part of the payoff can be evaluated as follows: first of all, let us ignore the coefficient $252/n$, i.e. we consider only

$$\mathrm{RV}_f(T) := \sum_{i=1}^{n} f(S_{t_{i-1}}) \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 .$$

As before, assume absence of any credit risk and that X has no jumps. Also, chose a function F such that

$$F''(s) = \frac{f(s)}{s^2} .$$

We can then write

$$\begin{aligned} \mathbb{E}\left[\,\mathrm{RV}_f(T)\,\right] &= \mathbb{E}\left[\int_0^T \frac{f(S_{t-})}{S_{t-}^2} d\langle S \rangle_t \right] \\ &= 2\mathbb{E}\left[\, F(S_T) - F(S_0)\,\right] - 2\mathbb{E}\left[\int_0^T F'(S_{t-})\, dS_t \right] - 2 \sum_{j:\tau_j \le T} D_j^f(S_{\tau_j}) \end{aligned}$$

with a "dividend adjustment" term due to (37),

$$\begin{aligned} D_j^f(s) \quad := \quad & \left\{ F(s) - F\left(\frac{s+\alpha_j}{1-\beta_j} \right) \right\} \\ & - F'\left(\frac{s+\alpha_j}{1-\beta_j} \right) \left(\frac{s\beta_j + \alpha_j}{1-\beta_j} \right) \\ & - \frac{1}{2} f\left(\frac{s+\alpha_j}{1-\beta_j} \right) \left(\frac{s\,\beta_j + \alpha_j}{s+\alpha_j} \right)^2 . \end{aligned}$$

For the example of a corridor (or conditional) variance swap, we have $f(s) := 1_{s>B}$, ie. $F''(s) := 1_{s>B}/s^2$, which gives

$$F'(s) = \left(\frac{1}{B} - \frac{1}{s} \right)^+ \quad \text{and} \quad F(s) = \frac{1}{B}\Big(s - B \Big)^+ - \left(\log \frac{s}{B} \right) 1_{s>B} .$$

In fact, these terms are also very intuitive: they essentially postulate that no delta-hedging or log-contract type European option position is required below the barrier B. Above the barrier, the hedge and the European position are essentially the (shifted) quantities of the variance swap hedge.

To evaluate $\mathbb{E}\left[\, F(S_T) - F(S_0)\,\right]$, we can employ approximation via Europeans as before. The delta-hedging term can be approximated as follows:

$$\mathbb{E}\left[\int_0^T F'(S_{t-})\, dS_t \right] \quad \approx \quad \mathbb{E}\left[\sum_{i=1}^{n} \left(F(S_{t_{i-1}}) S_{t_{i-1}} \right) \left(\frac{S_{t_i}}{S_{t_{i-1}}} - 1 \right) \right]$$

$$\approx \sum_{i=1}^{n} \mathbb{E}\left[F(S_{t_{i-1}}) S_{t_{i-1}} \right] \left(\frac{F_{t_i}}{F_{t_{i-1}}} - 1 \right) ,$$

i.e. it is again a matter of evaluating a string of European options. In the case of a corridor variance swap, we have to evaluate a sum of call options struck at the barrier,

$$\sum_{i=1}^{n} \frac{1}{B} \, \mathbb{E}\left[\left(S_{t_{i-1}} - B \right)^{+} \right] \left(\frac{F_{t_i}}{F_{t_{i-1}}} - 1 \right) .$$

Finally, the sum of the dividend terms $D_j^f(S_{\tau_j})$ is once again a simple strip of European options.

The other payoff variants are equally straight forward.

Remark 5.7 *Since all payoffs of the type (39) can essentially be priced with European options, their price coincides with the price computed using an implied local volatility framework.*

6 Summary

We have discussed a simple and theoretically sound framework for handling cash dividends and credit risk in equity modeling, and we have shown that the proposed approach of writing the stock price process as

$$S_t = \left\{ (F_t^* - D_t) X_t + D_t \right\} 1_{t<\tau}$$

in terms of a "pure stock price" X is the *only* arbitrage-free way of doing so.

We have shown the impact of our approach on the pricing of European options, on implied volatility, how to derive the appropriate PDE and we have shown how to incorporate dividend effects in the pricing and risk-management of variance swaps and related products.

While our framework makes no claim of sufficiency for today's very complex and highly developed markets, it is clearly a step forward towards a sound integration of cash dividends into any risk management system. Its numerical closeness to the standard "drift-only" methods in equity pricing make it particularly appealing.

A Proofs

A.1 Derivation of the Forward Price Formula

It has been mentioned in section 2.1 that the forward of stock price under the setting there is given as (2) on page 5.

Let us fix the observation time t and the maturity of the forward $T > t$ and let ℓ such that $\tau_{\ell-1} \leq t < \tau_\ell$. If we purchase δ amounts of shares at time t then we will receive according to our assumptions $\delta S_t \mu_t \, dt$ in every interval dt

from the lending out the stock. Moreover, at any dividend date τ_j between t and including T, we receive first a proportional dividend of β_j and then a cash dividend of α_j. All of the above payments terminate if the stock defaults.

We handle the proportional receivables (borrow and proportional dividends) first. We observe that we can actually ignore the default event because strictly speaking we still "receive" the same proportional payments – it is just that the stock price is zero. To remove the random nature of the payments, we always re-invest them immediately into the stock, which means that by maturity, we will own a total of $\delta e^{\int_t^T \mu_s \, ds} / \prod_{j:t<\tau_j \le T}(1-\beta_j)$ shares. Since we are supposed to deliver exactly one share at maturity, we chose

$$\delta := e^{-\int_t^T \mu_s \, ds} \prod_{j:t<\tau_j \le T} (1-\beta_j) \ .$$

Next, we handle the future cash dividends. If at time τ_k with $t < \tau_k \le T$ the stock has not yet default, we will own

$$\delta_{\tau_k} := e^{-\int_t^{\tau_k} \mu_s \, ds} \prod_{j:\ell \le j<k} (1-\beta_j) \ .$$

shares (note that it is important here that the proportional dividends are paid first). For each of those shares, we will receive a cash dividend of α_k, i.e. the total amount earned from the cash part of the dividend is $\delta_{\tau_k}\alpha_k$. However, since these payments are subject to default risk, we can not simply account for those future payments today at time t by discounting them back to t – instead, we need to forward cell CDS to guarantee the credit risk (in other words, we sell defaultable bonds with a notional equal to the dividends we expect to receive). In formulas, that means discounting the dividends with both the standard rates and the hazard rate.

In total, the value of all forthcoming dividends up to T, seen at time t is

$$\sum_{j:t<\tau_j \le T} \delta_{\tau_j} \alpha_j e^{-\int_t^T (r_s+h_s) \, ds}$$

Hence, our total initial investment will be

$$\delta - \sum_{j:t<\tau_j \le T} \delta_{\tau_j} \alpha_j e^{-\int_t^T (r_s+h_s) \, ds} \ ,$$

the accrued value of which constitutes our fair strike at maturity T. This is formula (2).

Note that we do *not* accrue this value with the credit spread (only with the standard interest rates) because we will deliver the share to our counterparty even in the event of a default of the underlying company S prior to maturity - mathematically, the only risk we need to cover is the missing receipt of the cash dividends. An important consequence of this is that the fair forward for a given stock under credit risk is equal to the well-known Black&Scholes case as long as we consider all future dividends as proportional.

We will re-invest these proceeds at S_t into buying additional stock, which means that our holding at time T is $\delta e^{\int_t^T \mu_u \, du}$ shares.

A.2 Proof of Theorem 2.1

We aim to show theorem 2.1 on page 6 using the notations on page 6. We start with the following lemma:

LEMMA A.1 *Assume "no free lunch without vanishing risk" (NFLVR) as defined by Delbaen/Schachermayer [6]. Suppose that rates, repo, credit risk and proportional dividends are zero. In this case, the stock price price has the form*

$$S_t = (F_t - D_t)X_t + D_t \tag{40}$$

where

$$D_t = \sum_{j:\tau_j > t} \alpha_j$$

and

$$F_t = S_0 - \sum_{j:\tau_j \le t} \alpha_j \ .$$

Proof of the lemma– We first recall that the stock price will drop at each dividend date τ_j by the cash dividend amount α_j. Between dividend dates, the process S is a non-negative local martingale due to NFLVR according to [6]. For each t such that $\tau_\ell \le t < \tau_{\ell+1}$ for some ℓ this means that $S_t \ge D_t = \sum_{j:j>\ell} \alpha_j$.[16] Note that D_t is an absorbing boundary for S_t due to the local martingale property.

We define the adjust initial stock price

$$S_0^* := F_t - D_t = S_0 - \sum_{j=1}^{\infty} \alpha_j$$

as in equation (9) on page 7. This allows us to rewrite the claim (40) as

$$S_t = S_0^* X_t + D_t \ .$$

To prove that we can always write S in this form, define reversely

$$X_t := \frac{S_t - D_t}{S_0^*}$$

with the understanding $X_t \equiv 1$ for the case $S_0^* > 0$.

By construction, X_t is non-negative with $X_0 = 1$. We will prove that it is indeed a local martingale. The only contentious points are the dividend dates. Fix some ℓ. On the interval $(\tau_{\ell-1}, \tau_\ell)$, we have

$$X_t := \frac{S_t - D_{\tau_\ell - 1}}{S_0^*} = \frac{S_t - (D_{\tau_\ell} + \alpha_j)}{S_0^*} \ ,$$

[16] To see this, note that for each finite $m > 0$, we have $S_t \ge \sum_{j=\ell+1}^{\ell+m} \alpha_j$, i.e. S_t dominates a monotone series which in turn implies that it dominates its limit.

which is consistently defined over the dividend date τ_ℓ:

$$X_{\tau_\ell} = \frac{S_{\tau_\ell} - D_{\tau_\ell}}{S_0^*} = \frac{S_{\tau_\ell -} - (D_{\tau_\ell} + \alpha_j)}{S_0^*}$$

This means that if $\tau_{\ell-1} \le t < \tau_\ell$, then

$$\mathbb{E}\left[\, X_{\tau_\ell + \varepsilon} \,\middle|\, \mathcal{F}_t \,\right] = \mathbb{E}\left[\, X_{\tau_\ell} \,\middle|\, \mathcal{F}_t \,\right] = \mathbb{E}\left[\, X_{\tau_\ell -} \,\middle|\, \mathcal{F}_t \,\right] = X_t \ ,$$

which shows that X is indeed a local martingale. □

Proof of theorem 2.1– In the presence of interest rates, borrow and proportional dividends, we simply re-iterate the discussion above for a stock price process $\tilde{S}_t := S_t/R_t$ (with "growth-rate discount factor" R_t defined in equation (8) on page 7) which pays dividends $\alpha_j^* := \alpha_j / R_{\tau_j}$. This gives

$$S_t = (F_t - D_t)X_t + D_t$$

for the stock price without default.

To also take into account credit risk, we multiply the non-default case with $\mathrm{SV}(0,t) \mathit{1}_{t>\tau}$, which leads to the statement in theorem 2.1. □

B Basics

B.1 Ito for jump processes

Let X and Y be two random cadlag processes with finite activity jumps (for example, a diffusion mixed with a Poisson-process). Denote by X^c and Y^c the continuous versions of X and Y, respectively, i.e. $dX^c = dX_{t-}$. Then,

$$\begin{aligned} dF(t,X_t,Y_t) &= \partial_t F(t,X_{t-},Y_{t-})\,dt \\ &\quad + \partial_x F(t,X_{t-},Y_{t-})\,dX_t^c + \partial_y F(t,X_{t-},Y_{t-})\,dY_t^c \\ &\quad + \frac{1}{2}\partial_{xx} F(t,X_{t-},Y_{t-})\,d\langle X^c\rangle_t + \frac{1}{2}\partial_{yy} F(t,X_{t-},Y_{t-})\,d\langle Y^c\rangle_t \\ &\quad + \partial_{xy} F(t,X_{t-},Y_{t-})\,d\langle Y^c, X^c\rangle_t \\ &\quad + \sum_t \Big\{ F(t,X_t,Y_t) - F(t,X_{t-},Y_{t-}) \Big\} \end{aligned}$$

If X and Y have jumps of infinite activity, then the above formula does not hold and the following reformulation has to be used

$$\begin{aligned} dF(t,X_t,Y_t) &= \partial_t F(t,X_{t-},Y_{t-})\,dt \\ &\quad + \partial_x F(t,X_{t-},Y_{t-})\,dX_t + \partial_y F(t,X_{t-},Y_{t-})\,dY_t \\ &\quad + \frac{1}{2}\partial_{xx} F(t,X_{t-},Y_{t-})\,d\langle X^c\rangle_t + \frac{1}{2}\partial_{yy} F(t,X_{t-},Y_{t-})\,d\langle Y^c\rangle_t \end{aligned}$$

$$+ \partial_{xy}F(t, X_{t-}, Y_{t-})\, d\langle Y^c, X^c \rangle_t$$
$$+ \sum_t \Big\{ F(t, X_t, Y_t) - F(t, X_{t-}, Y_{t-}) \Big\}$$
$$- \sum_t \Big\{ \partial_x F(t, X_{t-}, Y_{t-})\, (X_t - X_{t-}) + \partial_y F(t, X_{t-}, Y_{t-})\, (Y_t - Y_{t-}) \Big\}$$

B.2 Replication of Convex Payoffs with European Options

For any convex function f, let

$$h_1(x) := f(x_0) + w_1(x - x_0) \ , \quad \tilde{w}_1 := \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

such that $h_1 \geq f$ on $[x_0, x_1]$ and $h_1(x_1) = f(x_1)$. On $[x_1, x_2]$, set

$$h_2(x) := f(x_1) + \tilde{w}_2(x - x_1) = f(x_0) + \tilde{w}_1(x - x_0) + (\tilde{w}_2 - \tilde{w}_1)\,(x - x_1)$$

and

$$\tilde{w}_2 := \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Iteration and using

$$w_k := \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} - w_{k-1}$$

with $w_0 := 0$ yields that

$$h(x) := f(x_0) + \sum_{k=1}^{n} w_k (x - x_{k-1})^+$$

dominates f on $[x_0, x_n]$.

On the downside, we apply the same trick:

$$g(x) := f(x_0) + \tilde{w}_1'(x - x_0) = f(x_0) - \tilde{w}_1'(x_0 - x) \ , \quad \tilde{w}_1' := \frac{f(y_1) - f(x_0)}{y_1 - x_0}$$

such that

$$h(x) := f(x_0) - \sum_{k=1}^{m} w_k'(x_{k-1} - x)^+$$

with

$$w_k' := \frac{f(y_k) - f(y_{k-1})}{y_k - y_{k-1}} - w_{k-1}'$$

References

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