Expensive Martingales

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Abstract

We characterize strictly arbitrage-free markets of European options where only a discrete set of options is traded. We then construct martingales which reprice all given options and which are "most expensive" among all martingales with this property.

We also present algorithms to adjust real life market data and to construct expensive martingales while taking into account additional "weak" information: Estimated prices of more exotic products such as, for example, forward started options.

1 Introduction

It is well known that the implied volatility surface of a typical market does not exhibit just time-dependent implied volatilities, and that there is therefore a requirement to model processes which go beyond the classical Black, Scholes & Merton approach.

There are various attempts to find a process which can be calibrated to the market and which is then able to reproduce these observed option prices.

The first group of models, are local volatility models as proposed by Dupire [6] or Derman/Kani [5]. They address this problem very elegantly if there is a continuum of option prices in strikes and maturities. However, since this is not the case in practise, these methods have to be combined either with numerical optimization (for example Klopfer et al [13] or Avellaneda et al [3]) or with interpolation, for which it is difficult to ensure absence of arbitrage.

Another approach is to model the stock price directly with a parameterized martingale model. The most popular stochastic volatility model is probably Heston [11] and the use of Levy processes in finance has been discussed, for example, by Overhaus et al [14]. However, to determine the parameter of the respective stock model, the process has to be calibrated once more to the discrete market data.

All of these models therefore need to be fitted in one way or the other to a discrete number observed option prices. We propose that these prices should first be checked for absence of arbitrage before the calibration is performed, and we show how such an algorithm works and how existing market data can be turned into an strictly arbitrage-free surface.

Moreover, we will then show how to construct a discrete state and discrete time martingale which reprices all options and which can clearly be identified as a “most expensive” martingale with this property. This is insofar an important benchmark as it shows upper arbitrage bounds for prices of convex European payoffs. Our martingale will be a point process which jumps only on the strikes of the options present in the market. From a technical point, our model is closest to the approach from Derman and Kani [5]. In fact, our model is in some sense a high dimensional generalization of their model where the “tree” only has nodes in as few states as possible.

Since our “most expensive” process is not uniquely defined, we then show how additional “weak information” in the form of approximate prices of further Exotic options can be incorporated into the model.

This article is structured as follows: First we present some notation, then we discuss the notion of “strict arbitrage-freeness”. In section three, we discuss how the theoretical results of Kellerer [12] can be used to obtain our expensive martingales, and how weak information can be incorporated. In the final fourth chapter we briefly review how this model can be used to price Exotic payoffs.

We conclude with a summary and remarks on the relation to further literature.

2 Setup

Assume that we are given a stock price S_0 in a market where the instantaneous interest rates $(r_t)_{t\geq 0}$ and the forward process $F = (F_t)_{t\geq 0}$ of the stock are deterministic. The discounted forward $e^{-\int_0^t r_s\, ds} F_t$ is then the price of a zero-strike call under the previous assumption. Since the forward process is deterministic, we can assume that $S_t = X_t F_t$, where X is a local martingale if there is “no free lunch without vanishing risk” (see Delbaen/Schachermayer [4]).

The payoff of a call with maturity T and strike K can then be rewritten as

$$e^{-\int_0^t r_s\,ds}(S_T - K)^+ = e^{-\int_0^t r_s\,ds} F_T \left(X_T - \frac{K}{F_T} \right)^+ \ . \tag{1}$$

Since we assume that F and r are deterministic and because a price system should be linear, we can without loss of generalization consider X instead of S. (Note, however, that we implicitly assumed that all dividends are proportional.)

Assumption 1 *The market interest rates are zero, there is no drift in the stock (which we will denote by X) and today's spot X_0 is one. This is no restriction.*

Equation (1) allows us to convert "real" market prices into our "flat" setting. The strikes for the options written on X are now given "relative to the forward" by $\{K/F_T\}_{K,T}$. Note that if the "real" market prices are quoted on a grid (ie, the same fixed cash strikes across maturities), the relative strikes will no longer be in such a grid.

2.1 The Market

Additional to X, we also assume that we are given m European call option prices $\mathbb{C} = (\mathbb{C}_i)_{i=1,\ldots,m}$ with maturities $0 < \tau_1 \leq \cdots \leq \tau_n < \infty$ and strikes $k_1, \ldots, k_m \in \mathbb{R}^+$ are known. (In this article, $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$.) We hence assume there are no spreads.

We will also need the set of all strikes at τ,

$$\mathcal{K}_\tau := \{\ k_i \in \mathcal{K}\ : \tau_i = \tau \}\ .$$

We will assume for all maturities $\tau \in \mathcal{T}$ that $0 \in \mathcal{K}_\tau$. In other words, we assume that the zero strike call prices are provided for all maturities.

Assumption 2 *For all $\tau \in \mathcal{T}$, $\mathbb{C}(\tau, 0) = 1$.*

Note that violation of this property does not imply presence of arbitrage since X might only be a local martingale. However, we are interested in constructing true martingales so we impose the above assumption.

Since the call strikes $\mathcal{K} := \bigcup_{\tau\in\mathcal{T}} \mathcal{K}_\tau$ are only a finite set, it is likely that $\mathbb{C}(\tau, \max \mathcal{K}_\tau) > 0$. In this case, there is usually some freedom left for the call prices above $\max \mathcal{K}_\tau$ (also see the proof of proposition 3.1). We hence introduce the "zero price strike":

Assumption 3 *We assume that there is some arbitrary but finite "zero price strike" $k^* \gg \max \mathcal{K}$ such that $\mathbb{C}(\tau, k^*) \equiv 0$ for all $\tau \in \mathcal{T}$.*

This "zero price strike" is a technical utility which we need to construct our martingales (see proposition 3.1 below). It is no restriction on the market data since it can be arbitrarily large. It does, however, enter in the notion of a "most expensive" martingale below.

Some notation

Given the zero strike price, let

$$\mathcal{K}^*_\tau := \mathcal{K}_\tau \cup \{k^*\} \ .$$

We also define the number of non-trivial strikes for a maturity τ as

$$d_\tau := |\mathcal{K}_\tau| - 1 \ .$$

We can then renumber the strikes in $\mathcal{K}^*_\tau$ such that

$$0 =: k^\tau_0 < \cdots < k^\tau_{d_\tau} < k^\tau_{d_\tau+1} := k^*$$

We will also write $k^\tau \in \mathbb{R}^{d_\tau+2}$ for the column vector with entries k^τ_j. For the elements of $\mathcal{T}$ we write

$$0 =: \tau^0 < \tau^1 < \cdots < \tau^n$$

Since the maturities τ^j are index by some $j \in \{0, \dots, n\}$ we may also use the index as a reference, ie $\mathcal{K}_j$ instead of $\mathcal{K}_\tau$. The use of the letter j is confined to denote an index $j \in \{0, \dots, n\}$.

Given a measure μ we will use the symbol X to denote the stock price process under this measure, ie

$$\mathbb{E}_\mu \left[F(X) \right] := \int F(x)\, \mu(dx) \ .$$

A measure with a discrete support $\{x_1, \dots, x_M\}$ can be written as

$$\mu(dx) = \sum_{i=1}^{M} \mu_i \, \delta_{x_i}(dx)$$

where δ_y denotes the dirac measure in y. The vector $(\mu_i)_{i=1,\dots,M}$ of real numbers identifies the measure μ and we will not distinguish between this vector and μ itself if the support $\{x_1, \dots, x_M\}$ of the measure is clear.

A prime $'$ will denote the transpose of a matrix or vector.

2.2 Strict Absence of Arbitrage

Let us now define what we mean with absence of arbitrage

DEFINITION 2.1 *The surface* $\mathbb{C}$ *is called* strictly arbitrage-free *if there exists a martingale* X *on some stochastic base* $\mathbb{W} := (\Omega, \mathcal{A}, \mathbb{P}, \mathbb{F})$ *such that*

$$\mathbb{E}_\mathbb{P}\left[(X_{\tau_i} - k_i)^+ \right] = \mathbb{C}(\tau_i, k_i) \ .$$

We then say that X *(or its measure* $\mathbb{P}[X \in \cdot]$*)* reprices the market.

Note that we have used the same symbol X, which is a slight abuse of notation.

REMARK 2.1 *Note that the definition of "strict" arbitrage-freeness above is stronger than "No free lunch without vanishing risk", which in turn is equivalent to the existence of a* local *martingale (see Delbaen/Schachermayer [4]). This is reflected in our assumption 2.*

Note that this question is not purely academic, see for example Andersen et al [2].

The natural questions to ask are now: What are equivalent conditions on $\mathbb{C}$ such the market is strictly arbitrage-free ? And given a strictly arbitrage-free market $\mathbb{C}$, can we always *construct* a martingale which reprices the market ?

The first question is been answered by Kellerer [12]. The main contribution of this article is to clarify the construction of such a martingale.

3 Expensive Martingales

DEFINITION 3.1 *The* Balayage *order between measures μ and ν is defined as*

$$\mu \preceq \nu \qquad \text{iff} \qquad \int f(x)\,\mu(dx) \le \int f(x)\,\nu(dx)$$

for all convex functions f. We then say that ν is more expensive *than μ.*

LEMMA 3.1 *We have $\mu \preceq \nu$ if and only if*

$$\int (x-k)^+\,\mu(dx) \le \int (x-k)^+\,\nu(dx)$$

for all k.

See corollary 2.63 in Föllmer/Schied [8]. In his work [12], Kellerer showed that then

THEOREM 3.1 (Kellerer 1972) *Let $\mu = (\mu^t)_{t\in\mathcal{J}}$ be a set of probability measures with expectation 1, where $\mathcal{J} \subseteq \mathbb{R}_0^+$ is some Borel-set.*

Then, a Markov martingale $X = (X_t)_{t\in\mathcal{J}}$ with marginal distributions μ_t exists if and only if μ is in Balayage-order, that is

$$\mu^t \preceq \mu^u$$

for all $t < u$ with $t, u \in \mathcal{J}$.

Thanks to Jensen's inequality, the qualifier "Markov" is actually not necessary because the measures $\mu^t(dx) := \mathbb{P}[X_t \in dx]$ for some martingale X are naturally in Balayage-order.

COROLLARY 3.1 *Let $\mu = (\mu^t)_{t\in\mathcal{J}}$ be as above. Then, a martingale $X = (X_t)_{t\in\mathcal{J}}$ with marginal distributions μ^t exists if and only μ is in Balayage-order.*

Corollary 3.1 will be our main tool. Note that it is stronger than Dupire's [6] result since it is also applicable to non-continuous martingales.

3.1 Upper Pricing Measures

According theorem 3.1, absence of arbitrage is equivalent to $\mu = (\mu^\tau)_{\tau \in \mathcal{T}}$ being in Balayage-order. In our context, we do not have the full marginal distributions, but only a few discrete option prices. Clearly, the market $\mathbb{C}$ is strictly arbitrage-free in the sense introduced above if and only if we can construct a measure μ^τ for all maturities $\tau \in \mathcal{T}$ in our surface such that the set $\mu = (\mu^\tau)_{\tau \in \mathcal{T}}$ is in Balayage-order.

The first question is therefore how we can construct a measure μ^τ for just one given maturity.

To this end, define the first difference of the call price, which is the call spread between two strikes:

$$\Delta_i \mathbb{C}(\tau) := \frac{\mathbb{C}(\tau, k_{i+1}^\tau) - \mathbb{C}(\tau, k_i^\tau)}{k_{i+1}^\tau - k_i^\tau} \qquad i = 0, \ldots, d_\tau \ . \tag{2}$$

We also set $\Delta_{d_\tau + 1}\mathbb{C}(\tau) := 0$.

Definition 3.2 *We call a measure μ^τ* compatible *(in τ) if and only if*

$$\int (x - k^+)\, \mu^\tau(dx) = \mathbb{C}(\tau, k)$$

for all $k \in \mathcal{K}^\tau$.

Following Föllmer/Schied [8] section 7.4 (where the proposition is shown for the general continuous case, but without explicit construction of the upper pricing measure) we have:

Proposition 3.1 *A compatible measure μ^τ for τ exists if and only if the following conditions hold*

1. ***Positivity**: For all $k \in \mathcal{K}_\tau$,*
$$\mathbb{C}(\tau, k) \geq 0 \ . \tag{3}$$
2. ***Monotonicity**: For all $i = 0, \ldots, d_\tau - 1$,*
$$-1 \leq \Delta_i \mathbb{C}(\tau) \leq 0 \ . \tag{4}$$
3. ***Convexity**: For all $i = 1, \ldots, d_\tau - 1$,*
$$\Delta_{i-1}\mathbb{C}(\tau) \leq \Delta_i \mathbb{C}(\tau) \ . \tag{5}$$

In that case, we can define the upper pricing measure *for τ by*

$$\mu^\tau(dx) := \sum_{i=0}^{d_\tau + 1} \delta_{k_i}(dx)\mu_i^\tau \ . \tag{6}$$

with

$$\mu_i^\tau := \begin{cases} 1 - \Delta_0 \mathbb{C}(\tau) & (i = 0) \\ \Delta_i \mathbb{C}(\tau) - \Delta_{i-1}\mathbb{C}(\tau) & (i = 1, \ldots, d_\tau + 1) \end{cases}$$

Remark 3.1 *Instead of monotonicity, it is actually sufficient to assert* $-1 \leq \Delta_0\mathbb{C}(\tau)$ *and* $\Delta_{d_\tau}\mathbb{C}(\tau) \leq 0$ *(since monotonicity of the call prices then follows from convexity and the fact that* $\mathbb{C}(\tau,0) = 1$ *and* $\mathbb{C}(\tau, k^\tau_{d_\tau+1}) = 0$*).*

Also note that the above properties imply that $1 \geq \mathbb{C}(\tau,k) \geq (1-k)^+$.

Proof – We shall construct the requested measure. Let $d := d_\tau$. Note that by definition $k_{d+1} = k^*$ with $\mathbb{C}(\tau, k_{d+1}) := 0$.

Set

$$\mu(dx) := \sum_{i=0}^{d+1} \mu_i\, \delta_{k_i}(dx) \qquad \mu_i \in [0,1]\ . \tag{7}$$

We have to identify μ_i which sum up to 1 and which render μ compatible in τ. First, let

$$q_i := 1 + \Delta_i\mathbb{C}(\tau) \quad (i = 0,\dots,d).$$

This is the discrete equivalent of $\mathbb{P}[X_\tau \leq k] = 1 + \partial_k\mathbb{C}(\tau,k)$. From equations (4) and (5) we see that $0 \leq q_i \leq q_{i+1} \leq 1$.

Note that if $\Delta_d\mathbb{C}(\tau) < 0$ (which is the case if call with the highest initial strike has a non-zero price), then $q_d < 1$. We hence set $q_{d+1} := 1$, ie $\Delta_{d+1}\mathbb{C}(\tau) := 0$. That is, there is no probability mass beyond the "zero price strike" k_{d+1}.

Also note that we may have $q_0 > 0$ which reflects a possibility of default (since we construct a positive martingale, zero will be an absorbing state).

Now define

$$\mu_i := q_i - q_{i-1} \quad (i = 1,\dots,d+1) \tag{8}$$

and $\mu_0 := q_0$. All μ_i are then non-negative for $i = 0,\dots,d+1$ and they add up to one.

Now let $i \in \{-1, 0, \dots, d+1\}$. Then,

$$\begin{aligned}
\sum_{j=i+1}^{d+1} k_j\mu_j &= \sum_{j=i+1}^{d+1} k_j\,(q_j - q_{j-1}) \\
&= \sum_{j=i+1}^{d+1} k_j\,(\Delta_j\mathbb{C}(\tau) - \Delta_{j-1}\mathbb{C}(\tau)) \\
&= -\Delta_i\mathbb{C}(\tau)k_{i+1} + 0 + \sum_{j=i+1}^{d} (k_j - k_{j+1})\Delta_j\mathbb{C}(\tau) \\
&= -\Delta_i\mathbb{C}(\tau)k_{i+1} - \sum_{j=i+1}^{d} (\mathbb{C}(\tau,k_{j+1}) - \mathbb{C}(\tau,k_j)) \\
&= -\frac{\mathbb{C}(\tau,k_{i+1}) - \mathbb{C}(\tau,k_i)}{k_{i+1} - k_i}\, k_{i+1} + \mathbb{C}(\tau,k_{i+1}) + 0 \\
&= -\frac{\mathbb{C}(\tau,k_{i+1})k_i - \mathbb{C}(\tau,k_i)k_{i+1}}{k_{i+1} - k_i}
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
k_i \sum_{j=i+1}^{d+1} \mu_j = k_i \sum_{j=i+1}^{d+1} (q_j - q_{j-1}) &= k_i(q_{d+1} - q_i) \\
&= k_i(\Delta_{d+1}\mathbb{C}(\tau) - \Delta_i \mathbb{C}(\tau)) \\
&= -\frac{\mathbb{C}(\tau, k_{i+1}) - \mathbb{C}(\tau, k_i)}{k_{i+1} - k_i} k_i
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbb{E}_\mu \left[(X - k_i)^+ \right] &= \sum_{j=i+1}^{d+1} (k_j - k_i)\mu_j \\
&= -\frac{\mathbb{C}(\tau, k_{i+1})k_i - \mathbb{C}(\tau, k_i)k_{i+1}}{k_{i+1} - k_i} + \frac{\mathbb{C}(\tau, k_{i+1}) - \mathbb{C}(\tau, k_i)}{k_{i+1} - k_i} k_i \\
&= \mathbb{C}(\tau, k_i)
\end{aligned}
$$

Hence, the measure μ has expectation 1 (by setting $i = -1$) and reprices the market. $\square$

REMARK 3.2 *In the above construction of the measure μ, we used the "zero price strike" $k^* = k_{d+1}$ to account for the fact that a market price $\mathbb{C}(\tau, k^\tau_{d_\tau}) > 0$ leaves much room for possible call prices beyond k_d. As long as integrability and convexity is preserved a compatible measure could technically have an infinite support.*

The name "upper pricing measure" is justified by the following observation:

LEMMA 3.2 *Let μ^τ be the upper pricing measure for τ.*
Then μ^τ interpolates the call prices linearly.
Consequently,

1. *The measure μ dominates all compatible measures with support only on $[0, k^*]$ in the Balayage-order.*
2. *If ν is any compatible measure, than μ is more expensive for all calls with strikes $k \leq k_{d_\tau}$.*
3. *If ν is any measure with $\mathbb{E}_\nu \left[(X - k^\tau_i)^+ \right] \leq \mathbb{C}(\tau, k^\tau_i)$ for all $i = 0, \ldots, d_\tau + 1$, then μ^τ dominates ν.*

For the proof we will need the notion of the linear interpolation between to call prices. To this end, define

$$
\mathbb{C}^*(\tau, k) := \frac{k - k_i}{k_{i+1} - k_i} \mathbb{C}(\tau, k_{i+1}) + \frac{k_{i+1} - k}{k_{i+1} - k_i} \mathbb{C}(\tau, k_i) \quad k \in [k_i, k_{i+1}) \tag{9}
$$

and $\mathbb{C}^*(\tau, k) := 0$ for $k \geq k^*$.

Proof – First we show that $\mu = \mu^\tau$ interpolates the call prices linearly:

$$
\begin{aligned}
\mathbb{E}_\mu \left[(X-k)^+ \right] &= \sum_{j=i+1}^{d+1} (k_j - k)\, \mu_j \\
&= \sum_{j=i+1}^{d+1} ((k_j - k_i) + (k_i - k))\ \mu_j \\
&= \mathbb{C}(\tau, k_i) + (k_i - k) \sum_{j=i+1}^{d+1} \mu_j \\
&= \mathbb{C}(\tau, k_i) + \frac{k_i - k}{k_{i+1} - k_i} \left(\mathbb{C}(\tau, k_{i+1}) - \mathbb{C}(\tau, k_i) \right) \\
&= \mathbb{C}^*(\tau, k)
\end{aligned}
$$

Now we prove the last of the three statements in the lemma, which obviously implies the first. Number two is just a simple extension.

Let ν be a compatible measure.

For $k \in \mathcal{K}_\tau$ we have $\mathbb{E}_\nu \left[(X-k)^+) \right] \leq \mathbb{C}(\tau, k) = \mathbb{E}_\mu \left[(X-k)^+ \right]$. So let $k_i < k < k_{i+1}$ (we omit the explicit notion of τ). By convexity of the call price wrt strike,

$$
\mathbb{E}_\nu \left[(X-k)^+ \right] \leq \mathbb{C}^*(\tau, k) = \mathbb{E}_\mu \left[(X-k)^+ \right] \ . \tag{10}
$$

ie (10) applied to μ is an equality. Hence, μ dominates ν. $\square$

3.1.1 Call price functions

Corollary 3.1 makes it clear that the question on whether two measures are in Balayage-order is a matter of the relationships between the call prices. We therefore define

Definition 3.3 *A* call price function $c : \mathbb{R}^+ \to \mathbb{R}^+$ *is a function which can be represented as*

$$
c(k) := \int (x-k)^+ \, \nu(dx) \tag{11}
$$

for some measure ν with expectation 1 and support only on $\mathbb{R}_0^+$.

For a given measure ν, its *call price function is accordingly defined by (11).*

As a generalization of proposition 3.1 we have:

Proposition 3.2 *A function $c : \mathbb{R}_0^+ \mapsto [0, 1]$ is a call price function iff*

1. $c(0) = 1$, $\lim_{x \uparrow \infty} c(x) = 0$.
2. *c is positive.*

3. *c is decreasing with $c(\delta) - 1 \geq -\delta$ for all $\delta > 0$.*

4. *c is convex.*

Proof – Since c is convex and decreasing, its right-hand derivative c' exists and is right-continuous and non-decreasing. We also have that $c' \geq -1$ and $\lim_{x\uparrow\infty} c'(x) = 0$. Let $f(x) := 1 + c'(x)$, which is a positive, right-continuous and non-decreasing function and which therefore implies the existence of some positive σ-finite measure via $\nu[(a,b]] := f(b) - f(a)$ (cf. Aliprantis/Border [1] theorem 9.47 pg. 354). $\square$

We call one call price function c_2 *more expensive* than another call price function c_1 if and only if $c_2(x) \geq c_1(x)$. Thanks to corollary 3.1 that is equivalent to say that ν_2 is more expensive than ν_1.

So the upper pricing measure μ^τ is just more expensive than all other measures which are compatible with τ because it is the largest (since linear) convex interpolation between the discrete call prices $\mathbb{C}(\tau, \cdot)|_{\mathcal{K}_\tau}$.

In the sequel, we will need the lower call price function of two such functions

Definition 3.4 *Let c and e be two call price functions. Then,*

$$c \sqcap e := \sup\{\ h\ :\ h(x) \leq c(x) \wedge e(x) \text{ and } h \text{ is a call price function.}\} \tag{12}$$

is called the lower call price function *of c and e.*

For two measures μ and ν with call price functions c and e, we accordingly call the measure $\mu \sqcap \nu$ implied by $c \sqcap e$ the lower *measure of μ and ν.*

We have that $c \sqcap e(0) = 1$ and that $c \sqcap e$ is positive and convex (because the supremum of convex functions is convex). It also decreasing by the properties of the supremum. Hence the above definition makes sense because the fact that $(1-x)^+ \leq c(x) \wedge e(x)$ ensures that the set on the right hand side of (12) is not empty.

Also observe that $\mu \sqcap \nu \preceq \mu$.

Definition 3.5 *We call a set $c = (c^j)_{j=1,\ldots,n}$ of call price functions* strictly arbitrage-free *if the implied measures are strictly arbitrage-free.*

Notation 1 *We call $x \in \mathbb{R}^+$* extremal *point of c if*

$$\frac{c(x+\delta) - c(x)}{\delta} - \frac{c(x) - c(x-\delta)}{\delta} = \frac{c(x+\delta) - 2c(x) + c(x-\delta)}{\delta} > 0$$

for all almost all $\delta > 0$.

In case c is piecewise linear (as it will be in most of our applications), the extremal points are exactly those points where the slope of c changes.

3.2 Relative Upper Pricing Measures

The previous section showed that we can define an upper pricing measure for each maturity under the conditions of proposition 3.1. However, this did not take into account the term structure of the data we have.

For this reason, assume now that we have to maturities τ_1 and $\tau_2 > \tau_1$ with call strikes $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Recall that $\{0, k^*\} \subset \mathcal{K}_1 \cap \mathcal{K}_2$ by assumption. We assume that proposition 3.1 applies and that we can construct two upper pricing measures μ_1 and μ_2.

PROPOSITION 3.3 *If $\mathcal{K}_1 \supseteq \mathcal{K}_2$, then the measures μ_1 and μ_2 are in Balayage-order (ie, $\mathbb{C}$ is arbitrage-free) if and only if $\mathbb{C}(\tau_1, k) \leq \mathbb{C}(\tau_2, k)$ for all $k \in \mathcal{K}_2$.*

Proof – Apply third statement of lemma 3.2. □

In this situation, since the calls of the later maturity must dominate the prices for the earlier, we have to fit the convex τ_1-call price function below those later prices.

Now assume that $\mathcal{K}_1 \subset \mathcal{K}_2$. In this case, the situation is more involved: The upper pricing measure μ_1 might be too expensive between strikes k^2_{i-1}, k^2_{i+1} for τ_2. This can be seen from the following example:

Assume that for some i we have $\mathbb{C}(\tau_1, k^2_{i-1}) = \mathbb{C}(\tau_2, k^2_{i-1})$, $\mathbb{C}(\tau_1, k^2_{i+1}) = \mathbb{C}(\tau_2, k^2_{i+1})$, but no call price for a strike between k^2_{i-1} and k^2_{i+1} is available for τ_1 (call prices might well be flat in time may if there is no forward variance, for example during a weekend). Assume moreover, that $\mathbb{C}(\tau_2, \cdot)$ is not linear in this interval, ie that

$$\mathbb{C}(\tau_2, k^2_i) < \frac{k^2_i - k^2_{i-1}}{k^2_{i+1} - k^2_{i-1}}\ \mathbb{C}(\tau_2, k^2_{i+1}) + \frac{k^2_{i+1} - k^2_i}{k^2_{i+1} - k^2_{i-1}}\ \mathbb{C}(\tau_2, k^2_{i-1})$$

In that case the upper pricing measure of μ_1 will dominate μ_2 in this interval (because the call prices implied by μ_1 are the result of linear interpolation). But that does clearly *not* imply that there is no τ_1-compatible measure which is dominated by μ_2. Indeed, then the measure implied by additionally setting $\mathbb{C}(\tau_2, k^2_i) := \mathbb{C}(\tau_1, k^2_i)$ exists and is compatible with τ_1 if the market is arbitrage-free.

Let us formalize this idea:

DEFINITION 3.6 *Let $\mu = (\mu^j)_{j=1,\dots,n}$ be the set of upper pricing measures. The set of relative upper call price measures $\bar{\mu} = (\bar{\mu}^j)_{j=1,\dots,n}$ is then given by*

$$\begin{aligned} \bar{\mu}^n &:= \mu^n & (13) \\ \bar{\mu}^j &:= \mu^j \sqcap \bar{\mu}^{j+1} \qquad (j = n-1, \dots, 1)\ . & (14) \end{aligned}$$

By construction, the set $\bar{\mu}$ is in Balayage order, and each measure μ has support on $\bar{\mathcal{K}}_j := \bigcup_{i=j}^n \mathcal{K}_i$.

Lemma 3.3 *The market is strictly arbitrage-free if and only the* relative upper pricing measure *reprices the market.*

Proposition 3.4 *The relative upper pricing measures dominate the marginals of any martingale which reprices the market and whose marginals have support in* $[0, k^*]$. *For any martingale which reprices the market, the relative upper pricing measure is more expensive for all calls with strikes* $k \leq k^\tau_{d_\tau}$ *for* $\tau \in \mathcal{T}$.

By theorem 3.1, there exists a Markov-martingale X with marginals $\bar{\mu}$, which we will call an "expensive martingale". (Note that it is not unique).

Proof – [of lemma 3.3] First of all, if the relative upper pricing measures reprice the market, then the market is strictly arbitrage-free since $\bar{\mu}$ is in Balayage order.

Conversely, let $j := \max\{j : \bar{\mu}^j$ does not reprice the market$\} < n$. Set $T := \tau_j$, and denote the call price function of μ^T by c and the call price function of $\bar{\mu}^j$ by $\bar{c}$. Since μ^j reprices the market and since $\mu^j \succeq \bar{\mu}^j$, there exists $k \in \mathcal{K}_j$ such that $c(k) > \bar{c}(k)$. We have to show that this yields an arbitrage opportunity.

1. First assume that $k \in \mathcal{K}_j \cap \mathcal{K}_{j+1}$.

   Since $\bar{\mu}^j$ reprices the market, we have $\bar{c}^{j+1}(k) = \mathbb{C}(\tau^{j+1}, k)$ (where we denote by $\bar{c}^{j+1}$ the call price function $\bar{\mu}^{j+1}$). Given $\bar{c}(k) = c(k) \sqcap \bar{e}(k) < c(k) = \mathbb{C}(\tau^j, k)$ this implies an arbitrage opportunity at k.

2. So $k \in \mathcal{K}_j \cap \bar{\mathcal{K}}_{j+1} \setminus \mathcal{K}_{j+1}$.

   Hence, there is some $i := \min\{i > j : k \in \mathcal{K}_i\}$. If $\bar{c}^{j+1}(k) = \mathbb{C}(\tau^i, k)$, we can apply the argument of the point above. If $\bar{c}^{j+1}(k) < \mathbb{C}(\tau^i, k)$, then k is not extremal for $\bar{c}^{j+1}$.

   Now let k^- and k^+ the two extremal points $k^- < k < k^+$ of $\bar{c}^{j+1}$ which are closest to k (note that $\bar{c}^{j+1}$ is piecewise linear, so $k^\pm$ are well-defined). Then, $\bar{c}^{j+1}|_{[k^-,k^+]}$ is a linear function.

   Then there exists $i^-, i^+ > j$ such that $\bar{c}^{j+1}(k^\pm) = \mathbb{C}(\tau^{i^\pm}, k^\pm)$. But any valid call price function for τ^j must have $c(k^\pm) \leq c^{i^\pm}(k^\pm)$, which is not possible if $c(k)$ is above the linear interpolation $\bar{c}^{j+1}|_{[k^-,k^+]}(k)$ between $c^{i^-}(k^-)$ and $c^{i^+}(k^+)$

This ends the proof. $\square$

Proof – [of proposition 3.4] If a martingale X reprices the market, then the market is strictly arbitrage-free and $\bar{\mu}$ reprices the market, too. An argument similar to the above yields that $\bar{\mu}^\tau$ must then dominate the call prices of X_τ on the entire interval $[0, k^{d_\tau}]$. $\square$

The above discussion yields to equivalent conditions an strict arbitrage-freeness. We can employ the results now to implement two algorithms, the first of which tests whether a surface is arbitrage-free and the second which produces such an arbitrage-free surface "close" to given market data.

3.2.1 Test for Strict Arbitrage-freeness

Lemma 3.3 shows how to check mathematically whether a given call price surface $\mathbb{C}$ is arbitrage-free. From an implementation point of view, the following steps are to be performed:

1. Ensure the conditions of proposition 3.1 are satisfied for all maturities $\tau \in \mathcal{T}$. Otherwise, the respective marginal allows for arbitrage.

2. Construct the upper pricing measure μ^n for the last maturity τ^n. Let $\bar{\mathcal{K}}_n := \mathcal{K}_n^*$. Define $\bar{\mu}^n := \mu^n$ and let $\bar{c}^n$ be its call price function.

3. For each j,

   (a) Set $\bar{\mathcal{K}}_j := \mathcal{K}_j \cup \bar{\mathcal{K}}_{j+1}$.

   (b) Let $\bar{c}^j := c^j \sqcap \bar{c}^{j+1}$.
   This can be done by the following algorithm:

      i. Set $f(x) := c^j(x) \wedge \bar{c}^{j+1}(x)$ and denote by $1 = K_0 < \cdots < K_{m+1} = k^*$ be the strikes of $\bar{\mathcal{K}}_j$.

      ii. Define $h^0(x)$ as the line between $(1,1)$ and $(k^*,0)$.

      iii. For each strike $K_i, i = 1,\ldots,m$, now check whether $f(K_i) \geq h^{i-1}(K_i)$ and set $h^i := h^{i-1}$ in this case
      If $f(K_i) < h^{i-1}(K_i)$ find the strike K_ℓ with $\ell < i$ such that its left hand side derivative is less than

      $$\frac{f(K_i) - h^{i-1}(K_\ell)}{K_i - K_\ell}$$

      For this matter, the left hand side derivative at 0 is -1. Such a strike must exist because h^{i-1} is convex.
      Define the function h^i as h^{i-1} on $[0, K_\ell]$, and as linear interpolation between K_ℓ and K_i and K_i and $K_{m+1} = 1$, respectively.

      iv. We obtain $\bar{c}^j := h^m$.

4. Check if $\bar{c}^j(K) = \mathbb{C}(\tau^j, K)$ for all $K \in \mathcal{K}_j$. If not, there is a arbitrage opportunity.

Note that this algorithm also produces the relative upper pricing measures by means of their call prices.

3.2.2 How to Produce a strictly Arbitrage-free Surface

The above algorithm can also be implemented in an linear programming (LP) framework. To this end, we note that the conditions of proposition 3.1 are all linear conditions on the call prices $\mathbb{C}$.

We will now discuss how this observation can be used to produce an arbitrage-free surface from real life market data. Such data is prone to statistical difficulties. Moreover, trades for different options do not happen at exactly the same time, hence additional uncertainty is introduced in the market data by accumulating trade information. Consequently, minor violations of the arbitrage-conditions in a static snapshot of the market may actually not reflect real arbitrage-opportunities.

We assume that the transformation from real prices on S to prices of X using (1) has been performed.

Let us define as before the sets

$$\bar{\mathcal{K}}_j := \bigcup_{i=j}^{n} \mathcal{K}_i^*$$

and set $\delta_j := |\bar{\mathcal{K}}_j| - 2$. Define the vectors $K^j = (K_0^j, \ldots, K_{\delta_j}^j)'$ of strikes from $\bar{\mathcal{K}}_j$ and the weighting functions $w^j := (w_0^j, \ldots, w_{\delta_j}^j)'$ with $w_i^j := 1_{K_i^j \in \mathcal{K}_j}$. The weight w^j is therefore zero if there is no price $\mathbb{C}(\tau_j, K_i^j)$ available from the market for K_i^j with maturity τ_j. Note that the positive weights can be altered according to some user-choice.

Then define

$$s_i^j := \mathbb{C}^*(\tau^j, K_i^j)$$

where $\mathbb{C}^*$ is the linear interpolation as defined in (9).

We intend to compute a set $c = (c^j)_{j=1,\ldots,n}$ call price functions which is as close as possible to the initial market data, ie

$$\text{minimize } ||\vec{c} - \vec{s}||_w \tag{15}$$

where $||\cdot||_w$ is given in terms of some norm $||\cdot||$ using

$$||\vec{x}||_w := ||\vec{x} \star \vec{w}||$$

where $\star$ denotes component-wise multiplication. Here, we used $\vec{c}$ to denote the joint vector $\vec{c} = (c^{1\prime} \ldots c^{n\prime})'$. Clearly, we have to formulate conditions which constrain the minimization problem (15) to arbitrage-free call price functions c.

Ensuring absence of strict arbitrage in strike

Now fix some j and define the ratio

$$\alpha_i^j := \frac{1}{k_{i+1}^j - k_i^j}$$

for $i = 0, \ldots, d_j$. (When implementing this algorithm, we have to ensure that the strikes are sufficiently distant from each other to avoid numerical problems.)

In the light of remark 3.1, the conditions of proposition 3.1 translate into

1. Bounded parameters $1 = c_0^j \geq c_i^j \geq c_{d_j+1}^j = 0$ for $i = 1, \ldots, \delta_j$.
2. Bounded first derivatives,
$$-1 \leq \alpha_0^j c_1^j + \alpha_0^j c_0^j \quad \text{and} \quad \alpha_{d_j}^j c_{d_j+1}^j + \alpha_{d_j}^j c_{d_j}^j \leq 0$$
3. Convexity: For $i = 2, \ldots, \delta_j$:
$$\alpha_{i-1}^j c_i^j + \alpha_{i-1}^j c_{i-1}^j \leq \alpha_i^j c_{i+1}^j + \alpha_i^j c_i^j \ .$$
This can also be rewritten as the usual convexity condition
$$\alpha_i^j c_{i+1}^j - \left(\alpha_{i-1}^j + \alpha_i^j\right) c_i^j + \alpha_{i-1}^j c_{i-1}^j \geq 0 \ .$$

Remark 3.3 *In a similar approach to Härdle et al [10], we can reformulate the above conditions in terms of the first derivatives, too:*
$$\beta_i^j := \alpha_i^j c_{i+1}^j + \alpha_i^j c_i^j \ .$$

In any event all the above conditions are simple linear constraints on the call prices, which we can write as
$$A^j c^j \geq b^j \tag{16}$$
for a suitable matrix A^j and a vector b^j.

Strict arbitrage in time

Given now the matrices A^j, we also have to impose the condition that the call prices must be ordered in the Balayage-order. However, since the function c^{j+1} will be defined on all strikes on which c^j is defined, proposition 3.3 yields that it is sufficient to ensure that c^j is below the linear interpolation of c^{j+1}. Because of the convexity conditions on c^j, this is automatically satisfied if $c^j(k_i^{j+1}) \leq c^{j+1}(k_i^{j+1})$ for all $k_i^{j+1} \in \bar{\mathcal{K}}_{j+1} \subset \bar{\mathcal{K}}_j$.

Hence we find a (very sparse) matrix B^j such that
$$B^j \begin{pmatrix} c^j \\ c^{j+1} \end{pmatrix} \geq 0 \tag{17}$$
ensures that the call prices are increasing for all $j = 1, \ldots, n-1$.

Linear programming

In summary, we have found that the call price vector $\vec{c}$ must satisfy some linear constraints

$$U\vec{c} \geq v$$

to ensure that the resulting call price functions c are strictly arbitrage-free. This can now be used to compute a "closest" fit to the given market data by solving the program

$$\begin{aligned} \text{minimize } ||\vec{c} - \vec{s}||_w \\ U\vec{c} \geq v \end{aligned} \tag{18}$$

Note that this program will return the initial call prices $\mathbb{C}^*$ in case the market was strictly arbitrage-free from the start.

REMARK 3.4 *The above "full" linear program can be very extensive, if many maturities are involved. In this case, the program can also be executed "blockwise" from the back, bootstrapping the solution. This will not yield a true $||\cdot||_w$-optimal result but is considerably faster.*

REMARK 3.5 *If the sets $\mathcal{K}_j$ are very different from each other, many weights w_i^j will be zero and there is no unique solution to (18), which can produce unstable solutions.*

As a remedy, the weight of an artificial call price can be set to some $\varepsilon > 0$.

Note, however, that in our experience this routine does not generally yield an appropriate interpolation if the initial market data exhibits strong arbitrage. The resulting call price surface can be very different from a user's expectation and additional steps such as proper weighting must be taken to ensure that the surface meets the desired properties (such as tight fit around at-the-money, for example).

3.3 Expensive Martingales

We now assume that we are given a sequence of strictly arbitrage-free measures $\mu = (\mu^\tau)_{\tau\in\mathcal{T}}$ with masses only in the strikes $0 = k_0^\tau < \cdots < k_{d_\tau+1}^\tau := k^*$. We can now apply theorem 3.1 which asserts there must be a Markov process $X = (X_\tau)_{\tau\in\mathcal{T}}$ which reprices the market $\mathbb{C}$ implied by μ. As before, we use the indices $j = 1, \dots, n$ to refer to quantities related to the maturities $\tau^1, \dots, \tau^d$.

Let us denote the unit vector from $\mathbb{R}^{d_j+2}$ by 1^j for $j = 1, \dots, n$. We also use the notion μ^j for the d_j+2-dimensional column vector of point masses $\mu^j[k_i^j]$.

The transition-probabilities of X "from u to ℓ" are

$$\Pi_{u\ell}^j := \begin{cases} \mathbb{P}[\ X_j = k_\ell^j \mid X_{j-1} = k_u^{j-1}\ ] & \text{if } \mathbb{P}[X_{j-1} = k_u^{j-1}] > 0, \\ 0 & \text{otherwise.} \end{cases}$$

for $j = 1, \ldots, m$ (with, as usual, $X_j := X_{\tau_j}$ and $X_0 := 1$). This yields a matrix

$$\Pi^j := \left\{ \begin{array}{ccc} \Pi^j_{0,0} & \cdots & \Pi^j_{0,d_j+1} \\ \vdots & & \vdots \\ \Pi^j_{d_{j-1}+1,0} & \cdots & \Pi^j_{d_{j-1}+1,d_j+1} \end{array} \right\}$$

Hence a row represents the probabilities $\mathbb{P}[\ X_j \in dx \mid X_{j-1} = k^{j-1}_u\ ]$. We know that such a kernel exists, but how can we construct one? Let us formalize the notion of a stochastic kernel.

DEFINITION 3.7 *We call a Matrix $\Pi^j = (\Pi^j_{u\ell})$ with $d_{j-1} + 2$ rows and $d_j + 2$ columns a* Martingale-kernel *at $\tau_j \in \mathcal{T}$ iff*

1. *it is positive $\Pi_{u\ell} \geq 0$,*
2. *it is a conditional probability $\Pi 1^j = 1^{j-1}$ (all rows sum up to one) and*
3. *it has the martingale property $\Pi k^j = k^{j-1}$.*

We call such a kernel compatible with μ *iff it additionally*

4. *is a transition kernel for μ, ie $\Pi' \mu^{j-1} = \mu^j$.*

(The initial kernel Π^1 is just the transpose of μ^1.)

REMARK 3.6 *The set of compatible Martingale-kernels $\mathcal{P}$ is a convex set.*

DEFINITION 3.8 (Most expensive martingales) *Given Martingale-kernels $\Pi = (\Pi^j)_{j=1,\ldots,m}$, we call the Markov martingale X with $X_0 := 1$ and transition probabilities*

$$\mathbb{P}[\ X_j = k^j_\ell \mid X_{j-1} = k^{j-1}_u\ ] := \Pi^j_{u\ell}$$

the martingale of Π. *If Π is compatible with the relative upper pricing measures $\bar{\mu}$ of a market $\mathbb{C}$, then X is a* most expensive *martingale.*

3.3.1 Construction of a Transition Kernel

Now note that the properties of definition 3.7 are in fact all linear conditions on each matrix Π^j. Indeed, let us fix some j (the notion of which we will omit in this subsection) and consider the column vector of rows of Π^i,

$$\kappa := \big(\Pi_{0,0}, \ldots, \Pi_{0,d_j+1}; \Pi_{1,0}, \ldots, \Pi_{1,d_j+1}; \Pi_{d_{j-1}+1,0}, \ldots, \Pi_{d_{j-1}+1,d_j+1}\big)' \quad (19)$$

We have $\kappa \in \mathbb{R}^N$ with $N := (d_j + 2)(d_{j-1} + 2)$. The conditions *2* to *4* of definition 3.7 can be written as

$$\begin{array}{rcl} A\,\kappa & = & x \\ B\,\kappa & = & y\ . \end{array}$$

Here we use the $2(d_{j-1}+2)\times N$-matrix

$$A := \begin{pmatrix} 1^{j\prime} & 0^{j\prime} & & \dots & & & 0^{j\prime} \\ k^{j\prime} & 0^{j\prime} & & \dots & & & 0^{j\prime} \\ 0^{j\prime} & 1^{j\prime} & 0^{j\prime} & \dots & & & \vdots \\ 0^{j\prime} & k^{j\prime} & 0^{j\prime} & \dots & & & \vdots \\ \vdots & & & \dots & 0^{j\prime} & 1^{j\prime} & 0^{j\prime} \\ \vdots & & & \dots & 0^{j\prime} & k^{j\prime} & 0^{j\prime} \\ 0^{j\prime} & & & \dots & & 0^{j\prime} & 1^{j\prime} \\ 0^{j\prime} & & & \dots & & 0^{j\prime} & k^{j\prime} \end{pmatrix} \qquad x := \begin{pmatrix} 1 \\ k_0^{j-1} \\ 1 \\ k_1^{j-1} \\ \vdots \\ \vdots \\ 1 \\ k_{d_{j-1}+1}^{j-1} \end{pmatrix}$$

and the $(d_j+2)\times N$ matrix

$$B := \begin{pmatrix} v_0^{j-1} & 0 & 0 & \dots & 0 & v_1^{j-1} & 0 & \dots & & & 0 \\ 0 & v_0^{j-1} & 0 & \dots & 0 & 0 & v_1^{j-1} & 0 & \dots & & 0 \\ & & \ddots & & & & & \ddots & & & \\ & & & \ddots & & & & & & & \\ 0 & \dots & & 0 & v_0^{j-1} & 0 & \dots & & & 0 & v_{d_{j-1}+1}^{j-1} \end{pmatrix} \qquad y := \begin{pmatrix} \mu_0^j \\ \vdots \\ \mu_{d_j+1}^j \end{pmatrix}$$

Hence, define the $[(d_j+2)+2(d_{j-1}+2)]\times N$-matrix

$$M_1 := \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{and} \quad z_1 := \begin{pmatrix} x \\ y \end{pmatrix} \tag{20}$$

Now note that while the conditions encoded in $M_1\kappa = x_1$ admit at least one positive solution, they are not linearly independent. This is due to the fact that both μ^j and μ^{j-1} are probability measures and that both have unit expectation:

Since they are probability measures, we have

$$1^{j\prime}\mu^j = 1^{j-1\prime}\mu^{-1} = 1 \tag{21}$$

Now μ^j is given as

$$\mu^j = y = B\kappa$$

hence, say, $v_{d_{j-1}+1}^{j-1}$ can be expressed as a linear combination of v_u^{j-1} for $u = 0,\dots,d_{j-1}+1$.

The unit expectation of μ^j and μ^{j-1} on the other hand means

$$k^{j\prime}\mu^j = k^{j-1\prime}\mu^{-1} = 1$$

so we can express for example $v_{d_{j-1}}^{j-1}$ in terms of the other variables.

Consequently, we can reduce the system (20) to

$$M\kappa = z$$

by removing the last two rows of $(M_1|z_1)$.

This yields

Conclusion 3.1 *To find martingale kernels* Π *which are compatible with* μ*, we have to solve the linear programming "feasibility" problems*

$$\begin{cases} M^j \kappa = z^j \\ \kappa \geq 0 \end{cases} \tag{22}$$

for $M^j \in \mathbb{R}^{D^j \times N^j}$ *with* $D^j := (d_j + 2) + 2(d_{j-1} + 2) - 2$ *and* $N^j := (d_j + 2)(d_{j-1} + 2)$ *as above.*

This result is quite promising since linear programming problems can be solved efficiently and are well-studied. Given that the matrices M^j are very sparse, the solution of the LP problem above is usually solvable in reasonable time.

Remark 3.7 *For practical implementation, the matrices* M^j *can be further reduced by exploiting the following facts*

1. *For all states* k^j_ℓ *with* $\mu^j_\ell = 0$ *(ie states which have no mass in* τ_j*), the column* $(K^j_{u\ell})_{u=0,\dots,d^{j-1}+1}$ *can be ignored.*
2. *Equally, if* $\mu^{j-1}_u = 0$*, then the entire uth row can be omitted.*
3. *The states* 0 *and* k^* *are absorbing and the respective rows are therefore trivial.*

It also also possible to limit the range of the conditional probabilities by imposing additional conditions. However, it is not clear to us yet how this can be achieved while ensuring that a solution to the new problem still exists.

3.4 Incorporating Weak Information

The previous section has shown how we can construct a "most expensive" finite-state martingale if we are given a strictly arbitrage-free market. However, the mere fact that usually $D^j \ll N^j$ means that the system (22) has many solutions. Indeed, remark 3.6 shows that the set of solutions will be convex, hence as soon as there are just two possible solutions to (22), there will immediately be an infinite number of additional possibilities.

Also observe that most algorithms which solve linear-programming problems (see, for example Fang et al [7]) will usually find extremal solutions.

Now, the various kernels Π which satisfy (22) differ in the way they evaluate non-European functionals (while they agree for all European options). We can therefore chose to impose further constraints to identify a particular kernel of interest.

Remark 3.8 *Note that the problem here is in fact similar to a classical problem of pricing in an incomplete market, but under constraints. In fact, we ask to identify one pricing measure out of a set of martingale measures.*

The difference is that the various measures here do not need to be equivalent to each other.

3.4.1 Mean-Variance Pricing

One way to identify a unique solution to (22) is to impose an additional optimality criterion. In principle, this could be some conditional mean-variance criterion:

The martingale property yields $\Pi^j k^j = k^i_{j-1}$ for each conditional expectation. The conditional variance of the martingale of Π is therefore

$$\varsigma_i^j := \mathbb{E}\left[X_j^2 - \mathbb{E}\left[X_j \,\middle|\, X_{j-1} = k_i^{j-1} \right]^2 \,\middle|\, X_{j-1} = k_i^{j-1} \right] \; .$$

We can write this as

$$\varsigma_i^j = \Pi^j \, k^j I^j k^j - \left(k^i_{j-1}\right)^2$$

where I denotes the $d^j \times d^j$ unity matrix. This is a linear equation in Π. Hence, it is possible to minimize the variance over problem (22).

Other possibilities are possible. We want, however, concentrate on what we term “weak information”.

3.4.2 Weak Information

Let us fix some maturity τ_j. Assume that for this maturity, we have some “weak information”: Approximate prices of options on X_j and X_{j-1}. For example prices which are *probably* correct or for which we have a good estimate (for example, over-the-counter products which are not liquidly traded and with high spreads). Let $\mathcal{F}^j = \{\, f_i^j \,;\; i = 1, \ldots, z_j \,\}$ be some functions

$$f_i^j\left(x_j, x_{j-1}\right) \; .$$

For example, these could be some “forward start calls”

$$f_i^j\left(x_j, x_{j-1}\right) := \left(\frac{x_j}{x_{j-1}} - h_i \right)^+ 1_{x_{j-1}>0}$$

with strikes $h \in \{h_1, \ldots, h_{z_j}\}$. The price of such a function f under given pricing kernels $\Pi = (\Pi^j)_{j=1,\ldots,n}$ compatible with some measures $\mu = (\mu^j)_{j=1,\ldots,n}$ is then given as

$$\sum_{u=0}^{d_{j-1}+1} \mu_u^{j-1} \sum_{\ell=0}^{d_j+1} \Pi^j_{u\ell} \; f_{u\ell} \; .$$

where we used $f^j_{u\ell} := f(k^j_\ell, k^{j-1}_u)$. Let $\phi_{u\ell} := \mu_u^{j-1} f^j_{u\ell}$, then we can write the above equation in matrix notation conveniently as

$$\mu^{j-1\prime} \, \Pi^j \, f^j_{u\ell} = \Pi^{j\prime} \left(\mu^{j-1} \; f^j_{u\ell} \right) = \Pi^{j\prime} \phi_f \; .$$

Considering both $\Pi^j \equiv \kappa^j$ and $\phi_f \equiv \varphi_f$ as vectors, we see that the price of f under Π is given as

$$\pi^j(f) = \varphi_f' \kappa^j \; ,$$

which is once more just a linear equation in terms of κ^j. Hence, a set $\mathcal{F}^j$ of functions f for each maturity τ_j ($j > 1$) yields, for each j, an equation of the type

$$V^j \kappa^j = \pi^j \ .$$

Now assume we have "weak information" in the form of some estimated market prices $\tilde{\pi}^j$. Then, we can formulate

CONCLUSION 3.2 *The* weakly constrained *expensive martingale kernels* Π^j *are given as the solutions to the optimization problems*

$$\begin{cases} \textit{minimize} & ||V^j \kappa^j - \tilde{\pi}^j|| \\ \textit{such that} & M^j \kappa = z^j \\ & \kappa \geq 0 \end{cases} \tag{23}$$

for $M^j \in \mathbb{R}^{D^j \times N^j}$ *and* $V^j \in \mathbb{R}^{R^j \times N^j}$ *where* R^j *is the number of "weak information prices" at* τ_j*.*

Solutions to (23) can be found with straight-forward linear programming in case $||x|| := ||x||_\infty$ or $||x|| := ||x||_1$. In the more natural case $||x|| := ||x||_2$, we obtain a constrained linear least-squares programming problem, which can also be solved efficiently, see for example Fang et al [7]. The choice of a norm (which we could also equip with some additional weighting) indicates how we see our "weak information".

4 Applications

In this section, we assume that we are given a set of marginal distributions $\mu = (\mu^j)_{j=1,\dots,n}$ with compatible transition kernels $\Pi = (\Pi^j)_{j=1,\dots,n}$, ie we have $\mu^{j-1}\Pi^j = \mu^j$. We denote by $\mathcal{K}_j^{**} := \{\ k \in \bar{\mathcal{K}}_j\ :\ \mu^j[k] > 0\ \}$ the set of states at τ^j which have a non-trivial probability.

Given these data, we can construct a canonical martingale X on the following probability space: Let $\Omega_j := \{1\} \times \mathcal{K}_1^{**} \times \cdots \mathcal{K}_n^{**}$ and $\Omega := \Omega_n$. The associated filtration is given by the power sets $\mathcal{F}_j := \mathcal{P}(\Omega_n)$, the martingale X is the coordinate process and the measure $\mathbb{P}$ is defined by

$$\mathbb{E}_{\mathbb{P}}\left[\, F(X_1, \dots, X_n)\,\right] := \sum_{\ell_1=0}^{d^j} \mu^1_{\ell_1} \sum_{\ell_2=0}^{d^2} \Pi^2_{\ell_1,\ell_2} \cdots \sum_{\ell_n=0}^{d^n} \Pi^n_{\ell_{d-1},\ell_d}\ F(k^1_{\ell_1}, \dots, k^n_{\ell_n})$$

for bounded F.

4.1 Pricing

We now discuss how to price various options in this setup. We will consider both path-dependent and American style payoffs.

4.1.1 Path-dependent Payoffs

Clearly, European style options with payoffs

$$H(X_j)$$

can be priced using the simple formula

$$\pi(H) := \sum_{\ell=0}^{d_j+1} \mu_\ell^j \, H(k_\ell^j)$$

If we write $h = (h_0, \ldots, h_{d_j+1})'$ with $h_\ell := H(k_\ell^j)$, we can write

$$\pi(H) = h' \mu^j \ .$$

Now consider a payoff

$$H(X_1, \ldots, X_n) \ ,$$

for which we find the price

$$\pi(H) = \sum_{\ell_1=0}^{d_1+1} \mu_{\ell_1}^1 \sum_{\ell_2=0}^{d_2+1} \Pi_{\ell_1 \ell_2}^2 \ldots \sum_{\ell_n=0}^{d_n+1} \Pi_{\ell_{n-1}\ell_n}^n \, H(k_{\ell_1}^1, k_{\ell_2}^2, \ldots, k_{\ell_d}^d) \tag{24}$$

This can be computed by recursion.

This computation is growing polynomial: If we assume that $d_j \equiv d$ is constant, then the above formulation involves d^n multiplications. If d is very large, we can approximate the price also by using a discrete-state Monte-Carlo method. This is very efficient since drawing samples from a discrete distribution is quick and since the prices of European options can be used as control variates (for details on the use of control variates, see Glassermann [9]).

4.1.2 American options

To price simple American or Bermudan options with payoffs of the form

$$\sup{}_\sigma H(X_\sigma)$$

(where σ is a stopping time bounded by τ^d), we can employ the following algorithm:

1. Define the vector $f_i^n := H(k_i^n)$ for $i = 0, \ldots, d_n + 1$.
2. For all $j = n-1, \ldots, 0$ compute the *continuation value*

   $$c^j := \Pi^{j+1} f^{j+1}$$

   and define the new value vector

   $$f_i^j := H(k_i^j) \vee c_i^j \quad i = 0, \ldots, d_j + 1 \ .$$

3. The price is the scalar $\pi(H) := f_0^0$.

This application compute the *snell'sche envelope* for the pricing of H. See [8] chapter II.6 for details of the pricing of American options on discrete time steps.

Note that the effort to execute this program grows linearly in the number of maturities (if the number of strikes per maturity remains the same). There is one matrix multiplication involved per additional maturity.

4.1.3 Exotic American Options

The above pricing rule can be combined with the path-dependent pricing to allow for "Exotic" American payoffs. We call American options "exotic" if the exercise decision at time τ^j can depend on values of X in the past.

We see that the algorithm which computes the value of a path-dependent payoff using (24) recursively can simply also impose an additional optimality criterion at each step.

4.2 Relation to Other Models

The "most expensive martingale approach" presented here is closest related to Derman/Kani's Implied Tress [5]. Indeed, if we assume that all call prices are given, and if we define the sets $\mathcal{K}_j$ as being the nodes of the tree at time τ_j, then our model will yield the same pricing kernel as the implied tree by Derman/Kani.

In this sense, the "expensive martingales" here are a generalization of the tree idea. Our approach has the advantage that no call prices must be interpolated, if we have a sufficiently liquid market.

4.3 Related work and future research

In a recent paper, Härdle et al [10] have use a similar construction of our "Upper Pricing Measure" to investigate the statistics of the state price density implied from option prices.

Such investigation are crucial to assess whether the pricing scheme proposed here is suitable for indicative pricing, and it may well be developed further in direction of a "market model", which incorporates the dynamics found from such statistical observations into the expensive martingales here (so in a sense, we would obtain a randomized probability measure).

In another direction, it would be interesting to try to transfer the result obtained here to the continuous state case. In fact, we can see the development of discrete state kernels Π as a special case of writing Π in terms of some tensor basis over $\mathbb{R}_0^+ \times \mathbb{R}_0^+$.

Indeed, here we developed the kernel in terms of the basis given by the discretization

$$e_\ell^j(y) := \delta_{k_\ell^j}(y) \ .$$

Then we wrote (seing Π as a function $\Pi : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to [0,1]$)

$$\Pi^j(x,y) = \sum_{u=0,\ldots,d^{j-1}+1,\ell=0,\ldots,d^j+1} \Pi^j_{u\ell} e^{j-1}_u(x)\ e^j_\ell(y)$$

such that inductively

$$\mathbb{E}\left[\, F(X_{j-1}, X_j) \,\middle|\, X_{j-1} \,\right] := \int_0^\infty \Pi^j(X_{j-1}, dx_j)\ F(X_{j-1}, x_j)\ .$$

It is now natural to ask whether a similar approach can be performed with continuous bases $(e^j)_{j=1,\ldots,n}$ and also non-tensor bases on $\mathbb{R}_0^+ \times \mathbb{R}_0^+$.

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