Volatility Markets

Consistent modeling, hedging and practical implementation

vorgelegt von
Diplom-Mathematiker
Hans Bühler
Berlin

Fakultät II - Mathematik und Naturwissenschaften
der Technischen Universität Berlin
zur Erlangung des akademischen Grades
Doktor der Natuwissenschaften
Dr. rer. nat.

genehmigte Dissertation

Promotionsausschuss:
Vorsitzender: Prof. Dr. rer. nat. Stefan Felsner

Berichter/Gutachter: Prof. Dr. rer. nat. Alexander Schied
Berichter/Gutachter: Prof. Dr. rer. nat. Peter Imkeller
Gutachter: Prof. Dr. rer. nat. Josef Teichmann

Tag der wissenschaftlichen Aussprache:
26. Juni 2006

Berlin 2006,
D83

To my parents, for all their love and patience.

In Liebe, Hans.

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Chapter 1

Introduction

Ever since Black, Scholes and Merton published their famous articles [BS73] and [M73], huge markets of financial derivatives on a wide range of underlying economic quantities have developed. One of the most visible markets of underlyings is surely the equity market with index level and share price quotes being a common part of today's news programmes. Upon it rest deep exchange-based markets of "vanilla" derivatives on indices and single stocks.

This development of course changes the way over-the-counter products (those which are agreed upon on a case-by-case basis between the counter-parties) are evaluated and risk-managed. While Black and Scholes (BS) used only the underlying stock price and the bond to hedge a derivative in their model,[1] this cannot be justified anymore: their model is not able to capture what is today known as the "volatility skew", or "volatility smile", of the implied volatility of traded vanilla options. The root of the discrepancy is that volatility is not, as assumed in BS' model, a deterministic quantity. Rather, it is by itself stochastic.

The stochastic nature of the instantaneous variance of the stock price process is particular important if we want to price and hedge heavily volatility-dependent exotic options such as options on realized variance or cliquet-type products.[2] Such products cannot be priced correctly in the BS-model since their very risk lies in the movement of volatility (or variance, for that matter) itself.

Beyond Black-Scholes

There have been many approaches to remedy this problem: the most pragmatic idea is to infer an implied risk-neutral distribution from the observed market prices. To this end, Dupire [D96] has completely solved the problem of finding a one-factor diffusion which reprices a continuum of market prices. His *implied local volatility* is today a standard tool for evaluating exotic derivatives.

However, the resulting stock price dynamics are not overly realistic since the resulting diffusion is usually highly inhomogeneous in time. This implies that the model makes predictions about the future which are not matched by past market experience. Most notably, the implied volatility smile inside the model flattens out over time which is in contrast to the persistent presence of this phenomenon in reality. This in turn means that the dynamical behavior of the liquid options is not captured very well.

[1] In fact, the model is due to Samuelson [S65], but it is common to call it "Black&Scholes model".

[2] See section 1.1.2 for example payoffs.

Conceptually quite different from this fitting approach are *stochastic volatility* models. In these models, a parsimonious description of the dynamics of both the stock price and its instantaneous variance is the starting point. Such a model is based on “structural” assumptions on the underlying stock price. For example, Heston’s popular model [H93] assumes that the instantaneous variance of the stock price is a square-root diffusion whose increments are correlated to the increments of the return of the stock price. Other popular stochastic volatility models are Hagan et al. [HKLW02], Schoebel/Zhu [SZ99] and Fouque et al. [FPS00] to name but a few. In addition, there are also models which incorporate jump processes (see for example Merton [M76], Carr et al. [CCM98]) or mixtures of both concepts such as Bates [B96]. A good reference on models based on Lévy processes is Cont/Tankov [CT03].

Most of these structural models will lead per se to incomplete market models if only the stock and the cash bond are considered as tradable instruments. As a result, there is no unique fair price for most derivatives. To alleviate this problem in continuous models,[3] we have to extend the range of tradable instruments. Broadly speaking, each additional source of randomness requires an additional traded instrument to be able to hedge the resulting risk. This is called “completion of the market” (see also Davis [Da04]). However, it not clear which traded instruments we have to choose to complete our market.

Indeed, if we are to use the stock price together with a range of liquid reference options as hedging instruments, a more natural approach would be to model directly the evolution of the stock price and these reference options simultaneously. Such a framework has the advantage that the options are an integral part of the model and that the model yields hedging strategies directly in terms of the traded reference instruments.

The most prominent approach has been to model call and put prices via their implied volatilities. This has been undertaken by Brace et al. [BGKW01], Cont et al. [CFD02], Fengler et al. [FHM03] and Haffner [H04], among others. However, to our knowledge, none of theses *stochastic implied volatility* models is able to ensure the absence of arbitrage situations (such as negative prices for butterfly trades) throughout the life of the model.

For this reason, some authors have focused on the term-structure of implied volatilities for just one fixed cash strike. This approach has been pioneered by Schönbucher [S99] and has recently been put into a more general framework by Schweizer/Wissel [SW05] who also consider power-type payoffs. This approach is attractive for pricing strike-dependent options such as compound options. However, the dependency on a fixed cash strike also implies that if the market moves, the model’s fixed strike may drift too far out of the money to be suitable for hedging purposes. This is of particular concern if we want to price and hedge mainly volatility-dependent products such as options on realized variance or cliquets.

Consistent Modelling

In this thesis we propose to use *variance swaps* as reference instruments. Variance swaps essentially promise the payment of the *realized variance* of the returns of the underlying to the holder: their price is the market’s expectation of the realized variance of the returns of the stock up to the maturity of the contract. As such, variance swaps are inherently strike-independent and a natural candidate for volatility-hedging of volatility products: for contracts such as calls and puts on realized variance, they are the equivalent of the discounted forward on the underlying.

[3] Models with random jumps can generally not be completed using a finite number of additional instruments.

Before we take on the modeling of time-homogeneous Markov-models for variance swap markets (which have the advantage of preserving their principle structural properties over time), we first develop a theoretical framework of general variance swap term structure models. Indeed, we closely follow the ideas of Heath-Jarrow-Morton (HJM) [HJM92] for interest rates: the term structure of variance swaps will play the same role as the role played by term structure of zero bonds in HJM's framework. That means that instead of developing a model for the short variance directly (which is the subject of stochastic volatility models), we describe the dynamics of the entire implied variance swap price curve. From there, we construct compatible stock price processes and their corresponding implied short variance dynamics.

In a second step, we specialize the general framework to models which are driven by a finite-dimensional Markov process. The idea behind this type of models is that we first specify a functional form for the implied variance swap price curve and then drive the parameters of this curve in an arbitrage-free "consistent" way. (The notion of "consistency" originates again from the interest-rate world, where it was discussed first by Björk/Christenssen [BC99], and Björk/Svensson [BS01].) We will also discuss how a sensible instantaneous correlation structure between stock and variance curve can be implemented such that the resulting vector of stock price and curve parameters retains its Markov-property. These finite-dimensional processes are easier to handle and provide a "structural" access to a variance curve model. However, since it might be necessary to provide a perfect fit to the market under certain circumstances, we also show how we can move from a "structural" variance curve model to a "fitting"-type model, such as Dupire's stochastic volatility model [Du04].

Hedging

All our models will be developed directly under a local martingale measure. This approach ensures that there cannot be arbitrage in the model – but a second important question remains: the question of completeness. Having argued that variance swaps are a suitable hedging instrument in addition to the stock, we shall also provide the theoretical framework to assess when a variance curve model (or, in fact, any general Markov-driven model) generates a complete market. To this end, we show that if we only consider payoffs which are measurable with respect to the information generated by the traded assets (in opposition to the information generated by the background driving Brownian motion), then a financial model often allows the replication of a payoff, even if the volatility matrix of the tradable instruments with respect to the driving Brownian motion is singular: if a value function of an exotic product can be differentiated at least once in the parameter of the market instruments, then these derivatives provide as expected the desired hedging ratios. We will show that if the value function for each non-negative smooth payoff function whose derivatives have compact support is always continuously differentiable, then the entire market is shown to be complete. We will show that this is the case, for example, if the coefficients of the diffusion which drive the market instruments are continuously differentiable with locally Lipschitz derivatives. These ideas are put to use to obtain specific results in the context of variance swap curve models, where we need to impose an additional invertibility criterion on the variance curve functional in order to be able to back out the driving Markov factors by observing only a finite number of variance swaps. We also discuss briefly aspects of pricing if the asset in a market is a strict local martingale.

These results are all of theoretical nature. In practise, we cannot expect our model to fit perfectly. Indeed, a model will have to be calibrated to observed liquid option prices and the parameters which we obtain from a daily re-calibration will not be constant, as assumed by the model. Since the model itself does not provide a concept of hedging parameter-risk, it is common practise to perform "parameter-hedging": we are trying to find a portfolio of traded options such that our overall position is (reasonably) insensitive to changes in the parameters.

We will put this idea of parameter-hedging into a theoretical framework and will also present a new quick and efficient algorithm to obtain a "cheapest" portfolio of liquid options which both satisfies the desired accuracy of the parameter-hedge and which also takes into account real-life constraints such as transaction costs and transaction limits.

In a second part, we will then discuss the impact of the practice of re-calibration to the "meta-model" of the institution, in particular the question whether the real-life price processes which are the result of this recalibration remain local martingales. We will show that this is for example not the case if the speed of mean-reversion or the product of "volatility of variance" and "correlation" in Heston's model are not kept constant. Similar results are shown for other mean-reversion type models.

In the course of the discussion we also introduce what we will call "entropy swaps". They are closely related to another product, called "gamma swaps" or "weighted variance swaps". Appendix A.1.2 is devoted to the latter structures.

Practical Implementation

The third part of this thesis is the application of the first two parts: we discuss the implementation of a double mean-reverting variance curve model. It is shown that the proposed model is well-defined and that the associated stock price process is a true martingale. We then proceed and discuss a Monte-Carlo implementation which allows efficient evaluation of exotic products.

The resulting engine is finally used to calibrate the model in a multi-phase calibration routine. Even though the routine is based on Monte-Carlo, it is still relatively fast and yields good results for most major indices. We also employ efficient algorithms to detect arbitrage in European option markets and show how market data which violate arbitrage-conditions can be fixed.

Outline

This thesis is split into three consecutive parts: the first part is concerned with the development of variance swap curve models, the second part covers theoretical and practical issues of hedging and the third part discusses the implementation of a four-factor variance curve model.

Part I: Consistent Modeling

In section 2.2, we start by introducing general HJM-type variance curve models. We discuss basic properties and derive a Musiela-type parametrization. This approach has been introduced in [B06b]. We show that we can always construct an associated stock price which is at least a local martingale and mention a convenient method to determine whether it is a true martingale. The difference between fitting and structural models is also discussed.

In section 2.3, the framework is specialized to models where the variance curve is driven by a finite-dimensional homogeneous Markov-process: we develop the notion of a consistent pair (G, Z) of a variance curve functional G and a parameter process Z which drives this functional such that the resulting variance swap prices are local martingales. This is the fundamental idea of a Markov variance curve market model (it is also shown in remark 2.23 that implied local volatility can in fact be modeled within our framework). In section 2.4, we apply results from Björk/Christenssen [BC99], and Björk/Svensson [BS01] for the interest rate world and investigate when a Hilbert-space valued variance curve can be represented by a finite-dimensional realization.

Section 3 is devoted to examples: we present mean-reverting curve functionals (which lead to Heston-type models), double mean-reverting functionals (based upon which we develop a model in chapter 6) and other curve functionals which appear in the literature. We discuss two approaches which allow to turn a structural model into a fitting model in section 3.4.

Part II: Hedging

This part is divided into a first section on theory of complete markets and a second section which is concerned with the practise of parameter-hedging.

We start in section 4.1 by introducing the products we aim to replicate. Following the approach in [BT06], we then consider a general setting of a Markov-driven complete market in which we relax various standard assumptions in the literature (on the cost of stronger regularity assumptions). In particular, we will show that as long as the model "weakly preserves smoothness", the vector of traded instruments (with potentially an additional processes of finite variation) is extremal on its filtration, even if the volatility matrix is singular. This situation is not covered in most of the literature. To "preserve smoothness weakly" means that all non-negative smooth payout functions with compact support have value functions which are continuously differentiable in the price levels of the traded instruments. We point out that this holds for a diffusion, for example, if its drift and volatility coefficients are locally Lipschitz and continuously differentiable with locally Lipschitz derivatives.

The finding that such a condition is sufficient for market completeness is as important as it is intuitive: it shows that if we only consider payoffs which depend on the information generated by the observable tradable instruments (as opposed to the unobservable driving background Brownian motion), then we can replicate such payoffs with the tradable instruments as long as they are mildly well-behaved as specified above. Essentially, the result is that "delta hedging works" if the value function of a payoff is differentiable in the spot levels of the tradable instruments.

All this is then put into the framework of our variance swap curve models in section 4.2.3: an additional complication stems from the availability of an infinite number of variance swaps. We will give sufficient conditions under which it is possible to make use of only a finite number of variance swaps to hedge any exotic payoff. We also show how "variance swap deltas" can be computed in Markovian models.

We turn to practical issues in chapter 5: there, we will introduce the concepts of "calibration", "recalibration" and "parameter-hedging". We will put these ideas into a mathematical framework and will then discuss in section 7.3 an efficient algorithm which allows selecting a hedging portfolio from a large number of traded instruments under constraints. This algorithm, and the subsequent generation of compatible transition kernels (appendix D), has been presented in [B06a].

Afterwards, we turn to the theoretical implications of the practise of parameter-hedging: in section 5.3, we show that some of the parameters of models such as Heston or other mean-reverting models cannot be recalibrated if we want to avoid "dynamic arbitrage". This has also been highlighted in [B06b]. Additionally, we introduce in section 5.3.1 "entropy swaps" which allow us to extend the results for Heston's model. Appendix A.1.2 discusses a closely related product, called "gamma swaps". These products are discussed in more detail in [BBFJLO06].

Part III: Practical Implementation

This part of the thesis shows how a Markov variance curve market model can be implemented.

We present the four-factor model which provided good fits to observed market data and discuss its mathematical properties in section 6.1.1: we show that its SDE has a unique solution and that the stock price is a true martingale. The following section, 6.2, is then devoted to the implementation of an efficient unbiased Monte-Carlo Milstein scheme which can be used to price exotic payoffs. We also show how European options can be priced particularly efficiently. This extends the discussion of this model in [BBFJLO06].

These pricing methods are then used to calibrate the model. The calibration is performed in several steps: first, the market data of European options is checked for arbitrage and, if necessary, corrected (we present efficient algorithms for this purpose). In a next step, we calibrate the states of the model from the observed variance swaps prices. The remaining parameters are then calibrated using the European option prices.

We present example calibrations and discuss the behavior of the model in a few applications before we conclude in section 8.

1.1 Basic Assumptions

Since we aim to develop a methodology to price and hedge strongly volatility-dependent products, we choose to simplify the situation by assuming that the prevailing interest rates are zero and that the stock has a constant forward of 1. It is shown in appendix A.2 that this simplification is essentially the same as assuming that the interest rates and the forward, including potential proportional dividends, are deterministic.

Moreover, we will assume:

Assumption 1 *The stock price process is continuous.*

1.1.1 Variance Swaps

A zero mean *variance swap* with maturity T is a contract which pays out the *realized variance* of the logarithmic total returns up to T in exchange for a fixed strike (we can assume without loss of generality that this strike is zero).

The annualized realized variance of a stock price process S for the period $[0,T]$ with business days $0 = t_0 < \ldots < t_n = T$ is usually defined as

$$\frac{d}{n}\sum_{i=1}^{n}\left(\log\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2 .$$

The constant d denotes the number of trading days per year and is usually fixed to 252 such that $d/n \approx 1/T$. An example term sheet of a variance swap can be found in appendix B.[4] A standard result (e.g. Protter [P04], pg. 66) gives that

$$\langle \log S\rangle_T = \lim_{n\uparrow\infty}\sum_{i=1}^{n}\left(\log\frac{S_{t_i^n}}{S_{t_{i-1}^n}}\right)^2 , \tag{1.1}$$

[4] In the presence of dividends, the returns of the stock are adjusted accordingly to eliminate the effect of the dividends; see appendix A.2.

where the limit is taken over a fixed sequence of refining subdivisions ($0 = t_0^n < \cdots < t_n^n = T$).[5] To ease the modeling of variance, we will ignore the deterministic scaling factor $1/T$ and we will assume that (1.1) holds. This approximation works very well for variance swaps, but care should be taken in practise if we price short dated non-linear payoffs on realized variance, cf. remark 7.26 on page 125. See also Barndorff-Nielsen et al. [BNGJPS04] for a discussion on the error of this approximation.

ASSUMPTION 2 *A variance swap with maturity T pays the realized quadratic variation $\langle \log S \rangle_T$ to the holder.*

A price at time t of a variance swap with maturity $T < \infty$ will be denoted by $V_t(T)$. We set $V_t(T) = V_T(T) = \langle \log S \rangle_T$ if $t > T$ for notational convenience.

The market convention of quoting a variance swap is not its mere price, $V_0(T)$. Rather, the market quotes its "variance volatility" (also called "VolSet") which is the strike K such that

$$\frac{1}{T}\langle \log S \rangle_T - K^2$$

has zero initial value (hence the name variance "swap").

DEFINITION 1.1 *We call*

$$K_0(T) := \sqrt{\frac{V_0(T)}{T}} \tag{1.2}$$

the variance swap volatility *of the variance swap with maturity T.*

These variance swaps will be the cornerstone of our investigation. We shall develop hedging strategies which involve dynamic hedging of an exotic payoff with such variance swaps. Such hedging strategies can only work if the underlying assets are liquid enough and if there is a well-developed market for them. Hence, let us make a third fundamental economic assumption:

ASSUMPTION 3 *A liquid[6] and frictionless[7] market of variance swaps on S exists for all maturities $T < \infty$. In particular, at any time t, there are variance swap prices $V_t(T)$ for $t < T < \infty$ available in the market.*

REMARK 1.2 *Assumption 3 is nowadays largely satisfied for the world's main indices such as SPX, NDX, STOXX50E, GDAXI, FTSE, N225 and so on,[8] but it should be noted that at the time of writing all those markets are broker markets.[9]*

However, we argue that this is not a fundamental problem because most investment banks will be able to quote an internal fair price with a very tight spread. Hence, the desk which is to run the risk management for, say, options on variance can use the variance swap desk's internal valuation for their risk management.

[5] I.e, $\lim_{n\uparrow\infty} \sup_{i=1,\dots,n} |t_i^n - t_{i-1}^n| = 0$.

[6] Trading in variance swaps is instant and not subject to transaction size limits.

[7] There are no transaction costs, taxes or bid/ask spreads.

[8] We refer to the indices via their Bloomberg codes.

[9] This means that transactions can not be made via an exchange and that bid/ask spreads remain relatively high: e.g on September 26th 2005, the spread on a very liquid STOXX50E December 2006 variance swap is around 0.5 volatility points compared with 0.25 volatility points on an ATM European option with the same maturity). Also, the ability to trade continuously is constrained as each transaction is executed on a case-by-case basis.

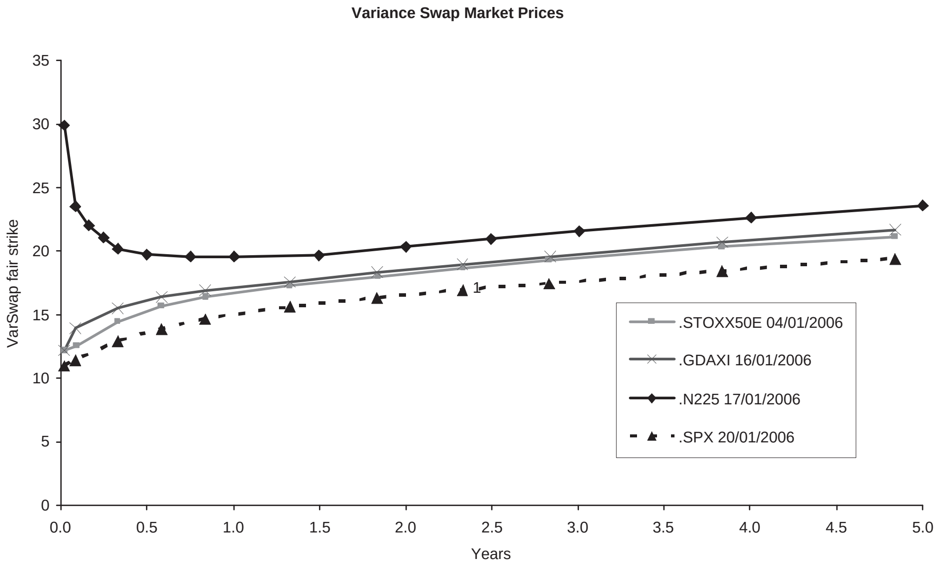


Figure 1.1: Example variance swap prices. The rices are quoted in "variance swap volatility" (1.2).

1.1.2 Options on Variance

Since we are going to model directly the prices of variance swaps, they become an *input* in our framework. The idea is to use a variance swap market model to price more exotic products. The most obvious class of structures which is suited for our approach is what we call "options on variance".

Here are a few examples of such products (precise definitions of the terms "options on variance" and "options on realized variance" can be found on page 60):

EXAMPLE 1.3 *Standard vanilla options on realized variance are calls and puts on realized variance,*[10]

$$\left(\frac{1}{T}\langle\log S\rangle_T - K^2\right)^+ \quad \textit{and} \quad \left(K^2 - \frac{1}{T}\langle\log S\rangle_T\right)^+ ,$$

or European options on realized volatility*,*

$$\left(\sqrt{\frac{1}{T}\langle\log S\rangle_T} - K\right)^+ \quad \textit{and} \quad \left(K - \sqrt{\frac{1}{T}\langle\log S\rangle_T}\right)^+ .$$

Other "options on variance" are options on forward variance swaps,

$$\left(\frac{V_T(T_2) - \langle\log S\rangle_T}{T_2 - T} - K^2\right)^+$$

where $T_2 > T$. This is an option on a variance swap with maturity T_2 which starts at time T.

Appendix B provides an example term sheet for a call on realized variance and a sheet for a volatility swap (i.e., a zero-strike call on realized volatility).

Such plain options on variance might be the most obvious application of a variance swap market model, but they are not the only products which can be priced and hedged within

[10]Strikes K are usually quoted in "volatility", hence the squared K in the payoffs to normalize them to (annualized) variance.

the framework presented here: since our approach also provides a consistent way to define a correlation structure between the stock and its *instantaneous variance* (see definition 2.22), such models are also well-suited to risk-manage classic "volatility" products if the correlation structure is well-defined.

Examples are:

- Forward-Started calls:
$$\left(\frac{S_{T_2}}{S_{T_1}} - k\right)^+$$
for $0 < T_1 < T_2$ and $k \geq 0$.
- Globally floored Cliquets:
$$\left\{\sum_{i=1}^{d} \min\left(\text{LocalCap}, \max\left(\frac{S_{T_i}}{S_{T_{i-1}}} - 1, \text{LocalFloor}\right)\right)\right\}^+ ,$$
for $0 = T_0 < \cdots < T_d$ and $\text{LocalFloor} \leq 0 \leq \text{LocalCap}$.
- Napoleons:
$$\left\{C + \min_{i=1,\ldots,d}\left(\frac{S_{T_i}}{S_{T_{i-1}}} - 1\right)\right\}^+ ,$$
for $0 = T_0 < \cdots < T_d$ and $C > 0$.
- Multiplicative Cliquets:
$$\left\{\left(\prod_{i=1}^{d} \max\left(\frac{S_{T_i}}{S_{T_{i-1}}}, 1\right)\right) - 1\right\}^+ ,$$
for $0 = T_0 < \cdots < T_d$.

A term sheet for a Napoleon structure can also be found in appendix B on page 147.

1.2 Mathematical Notation

For most of the discussion we will adapt the notation of Revuz/Yor [RY99].

Basics

We will make use of the standard notations $x \vee y := \max(x, y)$, $x \wedge y := \min(x, y)$ and $x^+ := x \vee 0$. The symbol $\mathbb{R}_{>0}$ denotes all strictly positive real numbers $x > 0$ while $\mathbb{R}_{\geq 0}$ is the set of all non-negative numbers. We will write both $A \subset B$ and $A \subseteq B$ to denote $x \in A \Rightarrow x \in B$ (the symbol $\subseteq$ is used to indicate it is *common* that $A = B$). If A is a strict subset of B, we will write $A \subsetneq B$. The transpose of a vector
$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}$$
is denoted by x'. Since most vectors are considered column vectors (it will be explicitly mentioned if they are row vectors), we omit the prime if written in text; i.e. $x = (x_1, \ldots, x_d)$ shall denote the same column vector as above. Moreover, we write $\mathbb{R}^{d\times m}$ for the space of matrices with d rows and m columns; for $M \in \mathbb{R}^{d\times m}$ we denote by M_i^j the element with the jth column and the ith row (see equation (1.3) below for an example).

Measurability and Integrability

The Lebesgue measure on $\mathbb{R}^d$ is denoted by λ^d and we set $\lambda := \lambda^1$.

Let $\mathcal{A}$ be a σ-algebra. We use $L^p(\mathcal{A}, \mathbb{P})$ to denote the $\mathcal{A}$-measurable random variables X such that $\mathbb{E}_{\mathbb{P}}[|X|^p] < \infty$ (the notion of the measure is omitted if $\mathbb{P}$ is clear from the context).

Given a topological space U, we denote by $\mathcal{B}(U)$ its Borel-σ-algebra. For the canonical Wiener space, $\mathcal{P}$ denotes the predictable σ-algebra on $C[0,\infty) \times \mathbb{R}_{\geq 0}$.

Let $X = (X_t)_{t\geq 0}$ be a stochastic process. We denote by $\mathbb{F}^X = (\mathcal{F}_t^X)_{t\geq 0}$ its complete right-continuous filtration. Note that any process X defined up to $T < \infty$ can be defined for $t > T$ as $X_t = X_T$. For a given filtration $\mathbb{F}$, an adapted process X and an $\mathbb{F}$-stopping time τ, the stopped process is defined as $X_t^\tau := X_{\tau\wedge t}$.

If $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is a second filtration, we say $\mathbb{G}$ is a *sub-filtration* of $\mathbb{F}$, denoted by $\mathbb{G} \subseteq \mathbb{F}$, iff $\mathcal{G}_t \subseteq \mathcal{F}_t$ for all t. For two σ-algebras $\mathcal{A}$ and $\mathcal{B}$, we also define $\mathcal{A} \bigvee \mathcal{B}$ as the joint σ-algebra.

Stochastic Integration

Let $\mathbb{G}$ be a complete and right-continuous sub-filtration of $\mathbb{F}$, and let $\mathbb{Q}$ be a measure on $\mathcal{F}_\infty$. A process $X = (X_t)_{t\geq 0}$ is called a $(\mathbb{G}, \mathbb{Q})$-*martingale* on the stochastic base $(\Omega, \mathcal{F}_\infty, \mathbb{F}, \mathbb{P})$ if $X_T \in L^1(\mathcal{G}_T, \mathbb{Q})$ for all finite T and $\mathbb{E}_{\mathbb{Q}}[\, X_T \mid \mathcal{G}_t\,] = X_t$ for all $t < T < \infty$. Note that we do not require $\lim_{t\uparrow\infty} X_t$ to exist or to be defined: we will consider martingales up to arbitrary but only finite T.

The space of all continuous $(\mathbb{G}, \mathbb{Q})$-martingales $X = (X_t)_{t\leq T}$ with horizon $T < \infty$ is denoted by $\mathcal{H}_T(\mathbb{G}, \mathbb{Q})$. We also use $\mathcal{H}_T^2(\mathbb{G}, \mathbb{Q})$ for all continuous square-integrable martingales and $\mathcal{H}_T^{\mathrm{loc}}(\mathbb{G}, \mathbb{Q})$ for all continuous local martingales, i.e. those processes $X = (X_t)_{t\geq 0}$ such that there exists an increasing sequence of stopping times $\tau_d \leq \tau_{d+1}$ with $\lim_{d\uparrow\infty} \tau_d = T$ such that X^{τ_d} is element of $\mathcal{H}_T^2(\mathbb{G}, \mathbb{Q})$ for each d. Note that the stopping times can be chosen in a way such that X^{τ_d} is bounded.

To ease notation, we will omit the notion of the measure or the σ-algebra if it is clear from the context.

For a d-dimensional martingale $X = (X^1, \ldots, X^d) \in \mathcal{H}_T^2(\mathbb{G}, \mathbb{P})$ we define the set of admissible integrands $L_T^2(X; \mathbb{G}, \mathbb{Q})$ as all $\mathbb{G}$-predictable processes $\varphi = (\varphi_t)_{t\in[0,T]}$ such that

$$\mathbb{E}_{\mathbb{Q}}\left[\sum_{i=1}^{d} \int_0^T \|\varphi_s^i\|_2^2 \, d\langle X^i\rangle_s\right] < \infty \ .$$

The space $L_T^{\mathrm{loc}}(X; \mathbb{G}, \mathbb{Q})$, on the other hand, is the space of integrands for the local martingale X, i.e. all $\mathbb{G}$-predictable process $\varphi = (\varphi_t)_{t\in[0,T]}$ such that

$$\mathbb{Q}\left[\sum_{i=1}^{d} \int_0^T \|\varphi_s^i\|_2^2 \, d\langle X^i\rangle_s < \infty\right] = 1 \ .$$

Note that this property is invariant under equivalent changes of measure.

For all of the symbols $\mathcal{H}_T^p$, $\mathcal{H}_T^{\mathrm{loc}}$, L_T^2 and L_T^{loc}, we drop the notion of T if the respective property holds for all finite T.

Notation of Stochastic Integrals

Let $\mu : \mathbb{R}^m \to \mathbb{R}^m$ and $\sigma : \mathbb{R}^m \to \mathbb{R}^{m\times d}$ be measurable functions. Assume X is a d-dimensional continuous semi-martingale in the sense of Revuz/Yor [RY99]. Let $Y_0 \in \mathbb{R}^m$ and assume that

$Y = (Y_t)_{t \geq 0}$ satisfies:

$$Y_t^i = Y_0^i + \int_0^t \mu_i(Y_s)\, ds + \sum_{j=1}^{d} \int_0^t \sigma_i^j(Y_s)\, dX_s^j \tag{1.3}$$

for $i = 1, \ldots, m$ (note the notation of the matrix $\sigma_i^j(y)$ according to our convention above).

We write this equation also as

$$Y_t = Y_0 + \int_0^t \mu(Y_s)\, ds + \sum_{j=1}^{d} \int_0^t \sigma^j(Y_s)\, dX_s^j$$

or, even more compact,

$$Y_t = Y_0 + \int_0^t \mu(Y_s)\, ds + \int_0^t \sigma(Y_s)\, dX_s \ .$$

Finally, we denote by $\mathcal{E}(X)$ the *Doléans-Dade exponential*

$$\mathcal{E}_t(X) := \exp\left\{ X_t - \frac{1}{2}\langle X \rangle_t \right\}$$

of a semi-martingale X.

Part I

Consistent Modelling

Chapter 2

Consistent Variance Curve Models

In this chapter, we introduce the theoretical framework for models which are designed to capture the market prices of variance swaps alongside the stock price. Our initial approach is very similar to the well-known Heath-Jarrow-Merton (HJM) approach to interest rate modeling [HJM92]. We will then specialize the general case in section 2.3 to finite-dimensionally parameterized models which are easier to handle in practise.

We will start with an overview in which we will also formulate the main problems (**P1**) to (**P3**) which this chapter addresses. Technical details and definitions will follow in section 2.2 page 21ff.

Examples are presented in chapter 3; issues of market completeness and practical implications of hedging are the subject of chapter 4, while chapter 5 will use the results of chapter 3 to show how recalibration of stochastic volatility models can lead to static arbitrage in the "meta-model" of the institution. In particular, it is shown that the speed of mean-reversion in mean-reverting variance curve models must be kept constant. Chapter 6 then presents the implementation of an example model.

The core theory discussed here has been presented first in [B06b]. The theory is also discussed in a more applied context in Bermudez/Buehler/Ferraris/Jordinson/Overhaus/Lamnouar [BBFJLO06], where additional examples and practical applications are presented.

2.1 Problem Statements and Overview

The most fundamental question when modeling variance swaps and the stock price is clearly absence of arbitrage:

Problem (**P1**)
Given today's variance swap prices $V_0(T)$ for all maturities $T \in [0, \infty)$, we want to model the price processes $V(T) = (V_t(T))_{t\in[0,\infty)}$ and the stock price S together*, such that the joint market with all variance swaps and the stock price itself is free of arbitrage.*

Apart from the additional presence of the stock price, this closely resembles the situation in Heath-Jarrow-Morton (HJM) interest rate theory where the aim is to construct arbitrage-free price processes of zero bonds. We carry this similarity further and introduce the *forward variance curve* $(v(T))_{T\geq 0}$ of the log-returns of S, defined as

$$v_t(T) := \partial_T V_t(T) \qquad T, t \geq 0$$

on some stochastic base $\mathbb{W} := (\Omega, \mathcal{F}_\infty, \mathbb{P}, \mathbb{F})$ which supports an extremal Brownian motion W.[1]

We then have a HJM-type result, namely that (under the assumptions of the next section) $v(T)$ must be a local martingale for each T and therefore has no drift. This will be carried out in section 2.1, where we will also introduce the "Musiela-parametrization" $\hat{v}$ of v in terms of a fixed *time-to-maturity* x,

$$\hat{v}_t(x) := v_t(x+t) \qquad x, t \geq 0 \ .$$

It is then also shown in theorem 2.13 that for all Brownian motions B on $\mathbb{W}$ the market of all variance swaps $(V(T))_{T\in[0,\infty)}$ and the B-"associated price process" S, defined by

$$\left.\begin{aligned} S_t &:= \mathcal{E}_t(X) \\ dX_t &:= \sqrt{\hat{v}_t(0)}\, dB_t \end{aligned}\right\} \tag{2.1}$$

is free of arbitrage because S is a local martingale. In such a case we call the curve v a *variance curve model*, and B has the intuitive meaning of a "correlation structure". We want to emphasize that these no-arbitrage-conditions are very straightforward to enforce, in remarkable contrast to the severe difficulties in this respect with the "stochastic implied volatility models" mentioned in the introduction.

Remark 2.1 *We want to stress that we do* not *attempt to develop a model to price variance swaps — on the contrary, we assume that their market prices are given; we want to make use of this information to construct a market model of variance.*

Finite-Dimensional Realizations

In practise we are interested in forward variance curves which are given as a functional of a finite-dimensional Markov-process: we aim to represent $\hat{v}$ as

$$\hat{v}_t(x) = G(Z_t; x) \tag{2.2}$$

where $G : \mathcal{Z} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ for $\mathcal{Z} \subseteq \mathbb{R}^m_{\geq 0}$ open is a suitable non-negative function and where Z is an $\mathcal{Z}$-valued Markov process which is a strong solution to an SDE

$$dZ_t = \mu(Z_t)dt + \sum_{j=1}^{d} \sigma^j(Z_t)dW_t^j \qquad Z_0 \in \mathcal{Z} \tag{2.3}$$

defined in terms of the d-dimensional standard Brownian motion W. A pair (G, Z) is called consistent iff (2.2) defines a variance curve model for all $Z_0 \in \mathcal{Z}$. This leads to the natural question:

Problem (**P2**)
When are a parameter process Z and a functional G consistent?

This will be addressed in section 2.3, and we will show in theorem 2.24 that consistency essentially implies

$$\partial_x G(z; x) = \mu(z)\, \partial_z G(z; x) + \frac{1}{2}\sigma^2(z)\, \partial_{zz} G(z) \ .$$

[1] For details and a precise setup, please refer to section 2.2.

In this case, if the "correlation structure" in (2.1) is given in terms of a measurable "correlation function" $\rho : \mathcal{Z} \times \mathbb{R}_{\geq 0} \to [-1, +1]^d$ such that $S = \mathcal{E}(X)$ satisfies

$$dS_t = \sqrt{\hat{v}_t(0)} \sum_{j=1}^{d} S_t \rho^j(Z_t, S_t)\, dW_t^j \ ,$$

then we call the model a *Markov variance curve model* (definition 2.22): the vector (Z, S) is Markov and we will see in part II of this thesis that (under regularity assumptions) these kinds of models are also extremal on their filtration (theorem 4.19), i.e. that they allow the computation of their hedging ratios by differentiating the value function of a payoff in the stock and state parameters (corollary 4.21).

These results are closely related to the concept of "finite-dimensional realizations" (FDR) for HJM interest rate models, as introduced by Björk/Christensen [BC99] and Björk/Svensson [BS01]: we say that "a variance curve model $\hat{v}$ admits an FDR", if for every $z \in \mathcal{Z}$ there exists a consistent pair (G, Z) such that $\hat{v}_t(\cdot) = G(Z_t; \cdot)$ up to a strictly positive stopping time. Note that we now understand $\hat{v}_t(\cdot)$ and $G(Z_t; \cdot)$ as *functions*, and therefore omit the argument x.

Problem (**P3**)
Given a family $\hat{v}$ and a smooth functional G, when will $\hat{v}$ admit an FDR in terms of G?

We will solve this problem locally in Section 2.4 by following closely ideas from Filipovic/Teichmann [FT04]: writing $\hat{v}$ as a solution to an $\mathcal{H}$-valued SDE in an Hilbert-space $\mathcal{H}$

$$d\hat{v}_t = \partial_x \hat{v}_t dt + \sum_{j=1}^{d} b_t^j(\hat{v}_t)\, dW_t^j \ , \tag{2.4}$$

we show in Theorem 2.29 that $\hat{v}$ stays locally in $G(\mathcal{Z}) \subset \mathrm{dom}(\partial_x)$ if

$$b^j(\hat{v}) \in T_{\hat{v}}\mathcal{G}$$

for $j = 0, \ldots, d$ and $\hat{v} \in \mathcal{G} \setminus \partial\mathcal{G}$. The first component b^0 is the Stratonovich-drift of $\hat{v}$,

$$b_t^0(\hat{v}) := \partial_x \hat{v}_t - \frac{1}{2} \sum_{j=1}^{d} Db^j(\hat{v})\ b^j(\hat{v})$$

(we also show the relevant conditions on the boundary of $\mathcal{G}$). Additionally, we prove that if $\hat{v}$ stays locally in $\mathcal{G}$ and if G is invertible, then it has a finite dimensional representation

$$\hat{v}_t = G(Z_t)$$

in terms of a (locally) consistent parameter process Z which is explicitly given in terms of b and G.

2.1.1 Review of the Stochastic Volatility Case

For illustration, we assume in this subsection that we are *given* a continuous stock price process as a positive continuous local martingale S on a stochastic base $\mathbb{W} = (\Omega, \mathcal{F}_\infty, \mathbb{F}, \mathbb{P})$ whose complete and right-continuous filtration $\mathbb{F}$ is generated by an d-dimensional Brownian motion $W = (W^1, \ldots, W^d)$. We also assume that its variance swap prices are finite.

This subsection is intended to build some understanding of the required properties of a variance swap model, but from a logical point of view it can be omitted and the reader may immediately proceed to section 2.2 on page 21.

PROPOSITION 2.2 *If S is a positive local martingale, we can write it as*

$$S_t = \mathcal{E}_t(X) \tag{2.5}$$

where

$$X_t = \int_0^t \sqrt{\zeta_s}\, dB_s \tag{2.6}$$

for some $\sqrt{\zeta} \in L^{loc}$ and a Brownian motion B. Moreover, $X \in \mathcal{H}^2$.

Proof – Since S is a positive local martingale, we can write $S_t = \mathcal{E}_t(X)$ for some continuous local martingale X (cf. [RY99] pg. 328, prop. 1.6).

Hence, there exists $z \in L^{\mathrm{loc}}(W)$ such that $X_t = \sum_{j=1}^d \int_0^t z_s^j \, dW_s^j$ and therefore $d\langle X\rangle_t = \zeta_t \, dt$ with $\zeta_t := \sum_{j=1}^d (z_t^j)^2$. Moreover, $\zeta_t^{-2} 1_{\zeta_t > 0}$ is a valid integrand for X since $\mathbb{E}[\int_0^t \zeta_s^{-1} 1_{\zeta_s > 0} \, d\langle X\rangle_s] = \mathbb{E}[\int_0^t 1_{\zeta_s>0} \, ds] \le t < \infty$. Therefore, we can define

$$B_t := \int_0^t \frac{1}{\sqrt{\zeta_s}} 1_{\zeta_s > 0} \, dX_t + \int_0^t 1_{\zeta_s = 0} \, dW_s^1 \ ,$$

compare also [RY99] pg.203.

We have $\langle B\rangle_t = \int_0^t \big(\zeta_s^{-2}\zeta_s^2 1_{\zeta_s>0} + 1_{\zeta_s=0}\big)\, ds = t$, and since B is clearly adapted and continuous, it is a Brownian motion with the required property (2.6). Since the variance swap prices for all finite T are finite by the initial assumptions, we have $\mathbb{E}[\int_0^T \zeta_s \, ds] < \infty$, i.e. $X \in \mathcal{H}^2$. □

Hence, any positive continuous stock price process can be written as a "stochastic volatility model" (such as the examples in chapter 3 or the model discussed in chapter 6)

$$\frac{dS_t}{S_t} = \sqrt{\zeta_t}\, dB_t \ .$$

Note, however, that the joint process (S, ζ) will generally not be Markov.

Variance Swaps

Since variance swaps are assumed to be tradable at any time t, the time-t price $V_t(T)$ is given as the expectation of the quadratic variation of $\log S$ under a martingale pricing measure $\mathbb{P}$:[2]

$$V_t(T) = \mathbb{E}_{\mathbb{P}}\left[\, \langle \log S\rangle_T \mid \mathcal{F}_t \,\right] = \mathbb{E}_{\mathbb{P}}\left[\int_0^T \zeta_s \, ds \,\middle|\, \mathcal{F}_t \right] \ . \tag{2.7}$$

It is clear from standard arbitrage-theory that the measure $\mathbb{P}$ is in general not unique, i.e. the price processes $V(T)$ of the variance swaps with maturities $T \ge 0$ are in general not determined by specifying the stock price process (S, ζ) alone.

It is therefore necessary to fix a pricing measure, which we will assume for the remainder of this subsection is $\mathbb{P}$. We want to stress that by constructing directly the variance swap prices (which is the subject of this thesis), the prices of variance swaps are given by the market and any

[2] Under each martingale pricing measure, the price process of each variance swap are by definition martingales.

pricing measure must have property (2.7). Chapter 4 is devoted to questions of completeness in variance curve models.

Given that $V(T)$ is a martingale and due to the extremality of W, we find some $b(T) \in L^{\mathrm{loc}}$ such that

$$V_t(T) = V_0(T) + \sum_{j=1}^{d} \int_0^t b_s^j(T)\, dW_s^j \ .$$

Remark 2.3 (Pricing Variance Swaps using European Options)
Neuberger [N92] has shown that the price of a variance swap in the present framework can be computed as

$$V_0(T) \quad = \quad 2\int_0^1 \frac{1}{K^2}\mathcal{P}_0(T,K)\, dK + 2\int_1^\infty \frac{1}{K^2}\mathcal{C}_0(T,K)\, dK$$

where $\mathcal{P}_0(T,K)$ and $\mathcal{C}_0(T,K)$ denote quoted put and call option prices with maturity T and strike K. Note that option prices for all strikes are needed for this formula, which can make this way of pricing variance swaps very sensitive to the specification of out-of-the-money implied volatilities, in particular those on the downside where the option weights are high.

The above formula is proved in appendix A.1 where we also discuss the impact of dividends and interest rates.

Forward Variance

By construction, the curve $V_t(\cdot)$ is at any time t absolutely continuous with respect to the Lebesgue measure λ, hence we can define λ-almost everywhere the derivative along T,

$$v_t(T) := \partial_T V_t(T) = \mathbb{E}\left[\, \zeta_T \mid \mathcal{F}_t \,\right] \tag{2.8}$$

which is called the fixed maturity *T-forward variance* seen at time t (note that $v_t(T)$ is well-defined for $T > t$). Note the conceptual similarity with the forward rate in interest rate modeling.

Proposition 2.4 (HJM-Condition for Forward Variance) *For all $T \geq 0$, the process $v(T) = (v_t(T))_{t\geq 0}$ defined by (2.8) is a martingale which can be written as*

$$v_t(T) = v_0(T) + \int_0^t \beta_s(T)\, dW_s \tag{2.9}$$

with $\beta(T) \equiv \partial_T b(T) \in L^{loc}$.[3]

2.2 General Variance Curve Models

We now introduce our variance curve models: We want to *specify* the forward variance price processes $v(T)$, which we imagine as the expected future instantaneous variance as in the previous

[3]The fact that $\beta(T) = \partial_T b(T)$ follows because for each T, (2.9) holds. Integration along T and exchanging integration gives that $\int_0^T v_t(u)\, du - \int_0^T v_0(u)\, du = \int_0^t\int_0^T \beta_s(u)\, du\, dW_s$, the right hand side of which is equivalent to $V_t(T) - V_0(T)$. The uniqueness of the martingale representation of $V(T)$ then shows that $\beta(T) := \partial_T b(T)$ in $L^{\mathrm{loc}}(W)$.

section. This process should allow us to construct a stock process such that the joint market of stock and variance swaps is arbitrage-free.

As before, we assume that we are given a stochastic base $\mathbb{W} = (\Omega, \mathcal{F}_\infty, \mathbb{F}, \mathbb{P})$ which supports an d-dimensional $\mathbb{P}$-Brownian motion W which is *extremal* on the complete and right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. This means that for any local martingale $X \in \mathcal{H}^{\text{loc}}$ there exists an $\varphi \in L^{\text{loc}}(W)$ such that

$$X_t = X_0 + \sum_{j=1}^{d} \int_0^t \varphi_u^j \, dW_u^j \ .$$

This property is also called the *predictable representation property*, or *PRP*. Finally, we also assume that $(\Omega, \mathcal{F}_\infty)$ is Polish (for example, if it is the standard Wiener space). This is required in proposition 2.10 below.

Recall that according to assumption 1, there are no interest rates and the forward process of the underlying stock price (which we have to model) is constant 1.

Definition 2.5 (Variance Curve Model) *We call a family $v = (v(T))_{T\geq 0}$ of processes $v(T) = (v_t(T))_{t\geq 0}$ a* Variance Curve Model *on $\mathbb{W}$ if:*

(a) For all $T < \infty$, $v(T)$ is a non-negative continuous local martingale with representation

$$dv_t(T) = \sum_{j=1}^{d} \beta_t^j(T) \, dW_t^j \tag{2.10}$$

for some $\beta(T) \in L^{loc}$ (this is the “HJM-condition” for variance curves).

(b) For all $T < \infty$, the initial variance swap prices are finite,

$$V_0(T) := \int_0^T v_0(x) \, dx < \infty \ . \tag{2.11}$$

(c) The process $v_\cdot(\cdot)$ is predictable (for example, if $v_t(\cdot)$ is left-continuous).

The family v is called a strong *variance curve model, if $v(T)$ is a martingale for all finite T.*

By proposition 2.4 it is clear that forward variance must be a local martingale. Finiteness of the variance swap prices is a very natural assumption if we want to use them as liquid instruments. Condition (c) is technical and used below to ensure that the short variance is well-defined.

Proposition 2.6 *Let v be a Variance Curve model. The variance swap price processes $V = (V(T))_{T\geq 0}$ given as*

$$V_t(T) := \int_0^T v_t(s) \, ds \tag{2.12}$$

are local martingales with dynamics

$$dV_t(T) = \sum_{j=1}^{d} b_t^j(T) \, dW_t^j$$

where $b_t^j(T) = \int_0^T \beta_t^j(s) \, ds$. They are true martingales if v is strong.

Proof – It is clear that $V(T)$ defined by (2.12) is adapted and therefore is a local martingale. If v is strong, then we have $V_t(T) = \mathbb{E}[\, V_T(T) \mid \mathcal{F}_t \,]$ and $V_0(T)$ by (2.11), hence $V(T)$ is true martingale for all finite T. The representation of V via b follows easily from (2.10) and the uniqueness of the representation of $V(T)$ with respect to W. □

Given a variance curve model, we call the positive process

$$\zeta_t := v_t(t) \tag{2.13}$$

the *short variance* of v. It is well defined by the requirements of definition 2.5. Given that $v(T)$ is a supermartingale (because it is a non-negative local martingale; cf. page 24), we have

$$\mathbb{E}\left[\int_0^T \zeta_s \, ds \right] = \int_0^T \mathbb{E}\left[\, v_s(s) \,\right] ds \le V_0(T) < \infty \ ,$$

and it follows that process $\sqrt{\zeta}$ is in L^2. This justifies the following definition:

Definition 2.7 (Associated Stock Price Process) *For any variance curve model v and an arbitrary real-valued Brownian motion B on $\mathbb{W}$, the B-*associated stock price process *is defined as the local martingale*

$$S_t := \mathcal{E}_t(X) \qquad \textit{with} \qquad X_t := \int_0^t \sqrt{\zeta_s} \, dB_s \ . \tag{2.14}$$

The process X is in $\mathcal{H}^2$ and if v is a strong variance curve model, then the variance swap prices on S are given as

$$\mathbb{E}\left[\, \langle \log S \rangle_T \mid \mathcal{F}_t \,\right] = \mathbb{E}\left[\, \langle X \rangle_T \mid \mathcal{F}_t \,\right] = V_t(T) \ ,$$

where V was defined in (2.12).

It follows then directly by construction

Theorem 2.8 (Variance Swap Market Model) *Let v be a variance curve model, B a Brownian motion and S its associated stock price process. Then, the joint market (S, V) is free of arbitrage and we call (S, V) a* variance swap market model.

We call it strong *if v is strong and if S is a true martingale.*

We see a very convenient property of the current model approach: once the variance curve model is fully specified by v_0 and the volatility structure β, an associated stock price process can easily be constructed to yield a full variance swap market model which is free of arbitrage.

Remark 2.9 (Interpretation of B) *Note that each B defined on the stochastic base $\mathbb{W}$ can be written as*

$$dB_t = \sum_{j=1}^d \rho_t^j \, dW_t^j$$

in terms of some stochastic "correlation vector" $\rho \in L^2(W)$ with $\rho_t \in [-1, +1]^d$ and $\|\rho_t\|_2 = 1$.

Since the Brownian motion B defines the "correlation structure" of S with its variance process, B has the intuitive meaning of a "skew parameter".

Note, however, while B can be chosen arbitrarily to yield a local martingale S, more care must be taken if S is required to be a true martingale. For example, we can try to satisfy Kazamaki's criterion, cf. Revuz/Yor [RY99] pg. 331, or Novikov's criterion, pg. 332. If the latter is satisfied, then S is a true martingale for all Brownian motions B. Another useful result bases on a nice argument from Sin [S98], which we will present in the next section.

2.2.1 The Martingale Property and Explosion of Variance

Let S be defined as in (2.14). Since it is a strictly positive local martingale, it is a *supermartingale*, i.e. $\mathbb{E}[S_T] \le \mathbb{E}[S_0] = 1$.[4] We now want to derive a condition under which S is a true martingale: we will show that S is a martingale if and only if the variance process under the measure associated with the numeraire S does not explode.

To this end, let now $\tau_n := \inf\{ t : \zeta_t \ge n \}$ and $\tau := \sup_n \tau_n$. The stopping time τ is called the *explosion time* of ζ and we say "ζ explodes under a measure $\mathbb{Q}$" if $\mathbb{Q}[\tau > T] > 0$ for some finite T. Note that ζ does not explode under $\mathbb{P}$ per construction, i.e. $\mathbb{P}[\tau \le T] = 0$.

We fix some finite T. For $n = 1, 2, \ldots$ Define the σ-algebra $\mathcal{G}_n := \mathcal{F}_{\tau_n \wedge T}$ and the discrete time process $D_n := S_{\tau_n \wedge T}$. Note that $D = (D_n)_n$ is a martingale on the filtration $\mathbb{G} = (\mathcal{G}_n)_n$, but that it is not necessarily uniformly integrable. However, on each $\mathcal{G}_n$, we can define a probability measure

$$\mathbb{P}^n[A] := \mathbb{E}_{\mathbb{P}}[D_n 1_A] \qquad A \in \mathcal{G}_n .$$

Since $(\Omega, \mathcal{F}_\infty)$ is Polish, so are $(\Omega, \mathcal{G}_n)$ and $(\Omega, \mathcal{G}_\infty)$ where $\mathcal{G}_\infty = \mathcal{F}_T$. Thanks to Kolmogorov's extension theorem (see, for example, Aliprantis/Border [AB99] corollary 14.27), there exists a measure $\mathbb{P}^S$ on $\mathcal{F}_T$ which is Kolmogorov consistent with the sequence $(\mathbb{P}^n)_{n\in\mathbb{N}}$, i.e.

$$\mathbb{P}^S[A] = \mathbb{P}^n[A] \quad \text{for all } A \in \mathcal{G}_n,$$

Intuitively, this is the measure where S is taken as a numeraire (in a localized sense). Its Lebesgue decomposition w.r.t. $\mathbb{P}$ is given as

$$\mathbb{P}^S[A] = \mathbb{E}[S_T 1_A] + \mathbb{P}^S[A \cap \{\tau \le T\}] \tag{2.15}$$

where the last component is singular to $\mathbb{P}$. In particular, (2.15) implies for $A = \Omega$ that

$$1 = \mathbb{E}[S_T] + \mathbb{P}^S[\tau \le T] ,$$

hence we obtain the following generalization of Sin's idea [S98]:

PROPOSITION 2.10 *The stock S is a martingale if and only if ζ does not explode under $\mathbb{P}^S$.*

We will make use of this proposition in chapter 6, section 6.1.1, to prove that the stock price of the model discussed there is indeed a true martingale. The interested reader finds a few more results on explosions in general diffusion models in chapter 10 of Stroock/Varadahan [SV79]. We will comment on the pricing and hedging in the case where S is a strictly local martingale in section 4.2.2.

2.2.2 Fixed Time-to-Maturity

In the sprit of Musiela's parametrization [M93] of forward rates, we now introduce the respective process for variance curve models:

DEFINITION 2.11 *We call*

$$\hat{v}_t(x) := v_t(t + x)$$

the fixed time-to-maturity forward variance*, and* $\hat{V}_t(x) := \int_0^x \hat{v}_t(s)\, ds$ *the* fixed time-to-maturity variance swap.

[4] Let $S_t^n := S_{\tau_n \wedge t}$ for a localizing sequence $(\tau_n)_n$ of stopping times. On $T \le \tau_n$, $S_T^n = S_T^{n+1} = S_T$, hence, by Fatou, $\mathbb{E}[S_T] = \mathbb{E}[\liminf_{n\uparrow\infty} S_T^n] \le \liminf_{n\uparrow\infty} \mathbb{E}[S_T^n] = S_0$.

Note that above definition is valid for each fixed t and almost all ω. To define a proper process $\hat{v}$, we have to impose some additional regularity on v.

PROPOSITION 2.12 *Let v be a variance curve model. Assume that v_0 is differentiable in T, that β in (2.10) is $\mathcal{B}[\mathbb{R}^d] \times \mathcal{P}$-measurable[5] and almost surely differentiable in T with*

$$\sqrt{\int_0^{T^*} \partial_\tau \beta_t^j(T)^2 \, dT} \in L^{loc}(W) \qquad \text{for } j = 1, \ldots, d \text{ and all } \tau < \infty \text{ and } T^* < \infty. \tag{2.16}$$

Then, $\partial_T v_t(T)$ coincides a.e. with $\partial_T v_t(T) = \partial_T v_0(T) + \sum_{j=1}^d \int_0^t \partial_T \beta_s^j(T) \, dW_s^j$ and the fixed time-to-maturity forward variance $\hat{v}(x)$ is of the form

$$\hat{v}_t(x) = \hat{v}_0(x) + \int_0^t \partial_x \hat{v}_s(x) \, ds + \sum_{j=1}^d \int_0^t \hat{\beta}_s^j(x) \, dW_s^j \tag{2.17}$$

where $\hat{\beta}_t^j(x) := \beta_t^j(t + x)$.

Proof – With the assumptions above, we have

$$\begin{aligned}
\hat{v}_t(x) &= v_t(t+x) \\
&\overset{(2.9)}{=} v_0(t+x) + \int_0^t \beta_u(t+x) \, dW_u \\
&= v_0(x) + \int_0^t \partial_T v_0(s+x) \, ds + \int_0^t \left\{ \beta_u(u+x) + \int_u^t \partial_T \beta_u(s+x) \, ds \right\} dW_u \\
&\overset{(*)}{=} v_0(x) + \int_0^t \left\{ \partial_T v_0(s+x) + \int_0^s \partial_T \beta_u(s+x) \, dW_u \right\} ds + \int_0^t \beta_u(u+x) \, dW_u \\
&= v_0(x) + \int_0^t \partial_T v_s(s+x) \, ds + \int_0^t \beta_u(u+x) \, dW_u \\
&= \hat{v}_0(x) + \int_0^t \partial_T \hat{v}_s(x) \, ds + \int_0^t \hat{\beta}_u(x) \, dW_u \,,
\end{aligned}$$

as claimed. Equation $(*)$ follows because of (2.16): property (2.16) basically ensures that $\int_0^s \partial_T \beta_u(T) \, dW_u$ is a local martingale (see, for example, Protter [P04] pg. 208). ☐

The reverse of the previous proposition constitutes the HJM-condition for the fixed time-to-maturity case: Assume we start with a family $\hat{v}$, when defines $v_t(T) := \hat{v}_t(T - t)$ a variance curve model?

THEOREM 2.13 (HJM-condition for Variance Curve Models) *Let $\hat{v} = (\hat{v}(x))_{x \geq 0}$ be a family of non-negative adapted processes $\hat{v}(x) = (\hat{v}_t(x))_{t \geq 0}$ such that:*

(a) The curve $\hat{v}(\cdot)$ is almost surely in C^1.

(b) The process $\hat{v}(x)$ has a representation

$$d\hat{v}_t(x) = \partial_x \hat{v}_t(x) \, dt + \sum_{j=1}^d \hat{\beta}_t^j(x) \, dW_t^j \ . \tag{2.18}$$

[5]Recall $\mathcal{P}$ was the predictable σ-algebra on $\Omega \times \mathbb{R}_{\geq 0}$.

(c) *The prices of variance swaps* $\hat{V}_0(x) := \int_0^x \hat{v}_0(s)\,ds$ *are finite for all* $x < \infty$.

(d) *The volatility coefficient* $\hat{\beta}$ *in (2.18) is* C^1 *and satisfies* $\sqrt{\int_0^{x^*} \partial_x \hat{\beta}_t^j(x)^2\,dx} \in L^{loc}$ *for all finite* x^*.

Then, the family $v = (v(T))_{T\in[0,\infty)}$ *given by*

$$v_t(T) := \begin{cases} \hat{v}_t(T-t) & t \le T \\ \hat{v}_T(0) & t > T \end{cases} \tag{2.19}$$

defines a variance curve model. If, moreover, $v(T)$ *is a true martingale for all* T*, then it is a strong variance curve model.*

Proof – We have to satisfy the conditions of definition 2.5. The finiteness of variance swap prices is satisfied by (b). Now assume v is defined by (2.19). As before,

$$dv_t(T) = d\hat{v}_t(T-t) = \sum_{j=1}^{d} \hat{\beta}_t^j(T-t)\,dW_t^j \ , \tag{2.20}$$

i.e. $v(T)$ is a local martingale. Let $\hat{z}_t(x) := \sum_{j=1}^n \hat{\beta}_t^j(x)\,dW_t^j$ and note that condition (d) above on $\hat{\beta}$ ensures that $\partial_x \hat{z}$ is well-defined. Hence, we can compute

$$\begin{aligned} v_T(T) - v_t(T) &\overset{(2.20)}{=} \sum_{j=1}^{d} \int_t^T \hat{\beta}_t^j(T-t)\,dW_t^j \\ &= \sum_{j=1}^{d} \int_t^T \left\{ \hat{\beta}_t^j(T) - \int_{T-t}^T \partial_x \hat{\beta}_t^j(y)\,dy \right\} dW_t^j \\ &= \hat{z}_T(T) - \hat{z}_t(T) - \int_{T-t}^T \left\{ \partial_x \hat{z}_T(y) - \partial_x \hat{z}_t(y) \right\} dy \\ &= \hat{z}_T(T-t) - \hat{z}_t(T-t) \ , \end{aligned}$$

so $v(T)$ is a local martingale. Finally, $\zeta_t := \hat{v}_t(0)$ is by construction well defined. □

This theorem allows us to specify $\hat{v}$ instead of v. We will therefore also refer to $\hat{v}$ as a "variance curve model" if it satisfies the conditions of theorem 2.13.

CONCLUSION 2.14 *Theorems 2.8 and 2.13 answer* (**P1**) *from the introduction.*

REMARK 2.15 *Despite the introduction of forward rates in terms of fixed-time-to-maturity by Musiela, it is more common in interest-rate theory to deal with fixed maturity objects because the maturities of underlying market instruments are typically fixed points in time (such as LIBOR rates and Swaps).*[6]

A variance curve, in contrast, is more naturally seen as a fixed time-to-maturity object, in particular given that the short end of the curve is the instantaneous variance of the log-price of the stock as seen in definition 2.7.[7]

[6] In a typical LIBOR rate model, the short rate is not modelled.

[7] For example, an option on realized variance (such as the call from example 1.3) is not an option on a variance swap.

2.2.3 Fitting the Market with Exponential Variance Curve Models

Let $\hat{f}_t(x)$ be the interest rate forward rate with time-to-maturity x observed at some time t. An advantage of the HJM-approach for interest-rates is that the current forward rate is given as

$$\hat{f}_t(x) = \hat{f}_0(x) + \int_0^t \Big(\partial_x \hat{f}_s(x) - \alpha_s(x) \Big)\, ds + \sum_{j=1}^d \int_0^t \hat{\beta}_s^j(x)\, dW_s^j$$

with HJM-drift $\alpha_t(x) := \sum_{j=1}^d \hat{\beta}_t^j(x) \int_0^x \hat{\beta}_t^j(y)\, dy$ so that the initial curve $\hat{f}_0$ can be estimated from market quotes without imposing additional constraints on the volatility structure $\hat{\beta}$.

In contrast, our specification of $\hat{v}$ must remain non-negative, which renders the specification of the volatility structure dependent on $\hat{v}_0$.

In the main part of this thesis we will deal with finite-dimensional realizations of $\hat{v}$, where this is not a concern (because we will write $\hat{v}$ in terms of a non-negative functional). However, if we were to work directly with $\hat{v}$, we might consider parameterizing it as

$$\hat{v}_t(x) = \hat{v}_0(x) e^{\hat{w}_t(x)} \ . \tag{2.21}$$

Proposition 2.16 *Equation (2.21) defines a variance curve model $\hat{w}$ iff $\hat{v}_0$ is in C^1 and if $\hat{w}$ with $\hat{w}_0 = 0$ has a representation*

$$d\hat{w}_t(x) = \left(\partial_x \hat{w}_t(x) - \frac{1}{2} \sum_{j=1}^d \hat{\gamma}_t^j(x)^2 \right) dt + \sum_{j=1}^d \hat{\gamma}_t^j(x)\, dW_t^j \tag{2.22}$$

for some $\gamma \in L^{loc}$ which is C^1.

One such model is presented in section 3.4.

As we mentioned before, ensuring that $v_t(T) = e^{\hat{w}_t(T-t)}$ is a true martingale is not trivial. However, if we want to allow arbitrary initial curves and be able to choose the volatility structure independently from the chosen initial curve, the approach above can be employed.

Remark 2.17 *In [Du04], Dupire discusses a model of the type above for a constant γ and a single driving Brownian motion, i.e. where $\hat{v}$ is log-normal. His article also contains details on hedging in such a framework. Also see example 3.10.*

Proof of the proposition– Let us first assume that

$$d\hat{w}_t(x) = \hat{\alpha}_t(x)\, dt + \sum_{j=1}^d \gamma_t^j(x)\, dW_t^j \ .$$

and that $\hat{v}$ defined in (2.21) is a variance curve model. This implies that v_0 is in C^1. Using Itô's formula and assuming $\hat{v}_0 \equiv 1$ for simplicity, we have

$$d\hat{v}_t(x) = \hat{v}_t(x) \left(\hat{\alpha}_t(x)\, dt + \sum_{j=1}^d \gamma_t^j(x)\, dW_t^j \right) + \frac{1}{2} \hat{v}_t(x) \left(\sum_{j=1}^d \gamma_t^j(x)^2 \right) dt \ .$$

Let $\beta^j := \hat{v}\gamma^j$, which is in C^1. Since $v_t(T) := \hat{v}_t(T-t)$ is a local martingale, we must have

$$\hat{\alpha}_t(x) + \frac{1}{2} \sum_{j=1}^d \gamma_t^j(x)^2 = \frac{\partial_x v_t(x)}{\hat{v}_t(x)} = \partial_x w_t(x) \ .$$

On the other hand, if $\hat{v}$ is defined by (2.21) for a process $\hat{w}$ satisfying (2.22) and $\hat{w}_0 = 0$, then another application of Itô's formula shows that $\hat{v}$ is a variance curve model. □

If we want to allow arbitrary initial curves and be able to choose the volatility structure independently from the chosen initial curve, this approach can be employed. Unfortunately, it does not allow $\hat{v}_t$ to be zero, hence models such as Heston's are not covered in this setting. It also forbids forward variances which are zero due to holidays or suspended trading.

However, we can extend this idea to fit an arbitrary variance curve model to observed market prices – this will discussed in section 3.4 on page 43.

Fitting the Market vs. Structural Models

Let us briefly comment on our decision to focus mainly on what we will call "structural" models as opposed to "fitting" models.

We call models with a parsimonious "functional" form (such as stochastic volatility models or the consistent variance curve models of the next section) "structural": these models try to describe the dynamics of the underlying and its volatility using an assumed dynamic (SDE) for the interaction of the various stochastic factors. In general, such models are given in terms of low-dimensional homogeneous Markov-processes.

On the other side of the spectrum, we have "fitting" models (chiefly Dupire's ground-braking *implied local volatility* [D96], but also his approach [Du04] cited above), which try to obtain as much relevant structure and dynamics from the observed market prices as possible. For example, the dynamical behavior of the stock in an implied local volatility model is completely determined by the initially observed set of option prices. In interest rates, a generic example is HJM's approach, but also Hull/White's "extended Vasiček" model [HW93].

The latter models provide a powerful pricing tool for structures which are "close" to the underlying market instruments. They are therefore very well suited for many standard applications.[8] Such models are usually given as non-homogeneous Markov-processes (or even non-Markov processes in the case of HJM-models).

However, in particular implied local volatility suffers from a lack of "predictive power": The future market data "scenarios" which are predicted by the model can differ widely from what users would accept as being realistic. This has been reported frequently by practitioners (see, for example, Overhaus [O05] or Hagan et al. [HKLW02]).

In contrast, while "structural" models will fit less well to today's observed market data, they make clearly defined predictions on the future shape of the market. For example, our variance curve models guarantee that the variance swap price curve is always of a certain shape. In the same vein, stochastic volatility models such as Heston [H93], and also our variance curve models, preserve the general shape of the implied volatility surface, because the Markov property of these models implies that this surface is a function of the (few) state parameters.

This matters if we want to risk-manage products which are not "close" to the calibration instruments. Consider for example the case of a forward started call spread with payoff

$$\left(\frac{S_{T_2}}{S_{T_1}} - K_1\right)^+ - \left(\frac{S_{T_2}}{S_{T_1}} - K_2\right)^+$$

[8] Note that since Dupire's local volatility fits perfectly the market of European options, it also perfectly reprices the variance swaps; see appendix A.1.

for $0 < T_1 < T_2$ and $K_1 < K_2$. At the time of writing, such options are not yet liquidly traded, so we have to use a financial model to evaluate and hedge them. However, as soon as the "reset date" T_1 is reached, the option turns into a standard call spread which is liquidly traded. It is therefore important that whatever model we use to compute the initial price, it makes reasonable predictions of the shape of the implied volatility surface (as a measure of option prices) at time T_1: in particular, the "skew" (i.e. the difference between the implied volatilities with strikes K_1 and K_2) needs to be realistic. This is achieved if we use structural models.

REMARK 2.18 *The above distinction between "structural" and "fitted" models is superficial. As remark 2.23 below shows, local volatility models are actually a sub-class of consistent variance curve models. In section 3.4 we will therefore discuss how to turn a "structural" variance curve model into a "fitting" model.*

2.3 Consistent Variance Curve Functionals

In the previous section, we have discussed variance curve models which were given in terms of general integrable processes. These have the aforementioned drawbacks: on one hand, it is very difficult to check whether a general model of the form (2.18) actually stays non-negative (this is particularly difficult for diffusions with values in Hilbert spaces, cf. equation (2.32) on page 35).

On the other hand, it is not clear how such models can be used in practise. Indeed, consider the situation in the reality of a trading floor: we do not actually see an infinite number of variance swap prices $(V_0(T))_{T\geq 0}$ in the market. Rather, a discrete set of swap prices will be interpolated by some functional which is parameterized by a finite-dimensional parameter vector.

Hence, we want to focus on variance curves which are given in terms of such finite-dimensionally parameterized *variance curve functionals.*

DEFINITION 2.19 (Variance Curve Functional) *A* Variance Curve Functional *is a non-negative $C^{2,2}$-function $G : (z;x) \in \mathcal{Z} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ such that $\int_0^T G(z;x)\,dx < \infty$ for all (z,T).*

The open subset $\mathcal{Z} \subset \mathbb{R}^m_{\geq 0}$ is called the parameter space *of G.*

Given a functional G, we now have to find a parameter process $Z = (Z_t)_{t\in[0,\infty)}$ such that

$$\hat{v}_t(x) := G(Z_t;x) \ , \quad x \geq 0 \ ,$$

forms a variance curve model. To avoid arbitrage, we need to meet the conditions of theorem 2.13. We want to focus on diffusions Z which are strong solutions of an SDE

$$dZ_t = \mu(Z_t)\,dt + \sum_{j=1}^d \sigma^j(Z_t)\,dW_t^j \tag{2.23}$$

with locally Lipschitz coefficients $\mu : \mathcal{Z} \mapsto \mathbb{R}^m$ and $\sigma^j : \mathcal{Z} \mapsto \mathbb{R}^m$ for $j = 1,\dots,d$ defined up to a strictly positive stopping time $\tau > 0$. The set of coefficients (μ,σ) which admit a unique strong solution for all $Z_0 \in \mathcal{Z}$ will be denoted by Ξ. We do not require that Z is confined to the set $\mathcal{Z}$; the question whether Z can leave $\mathcal{Z}$ is discussed in section 2.3.3 below. To ease notation we also refer to elements of Ξ as "processes" Z, even though they are actually families of processes (since they depend on the starting point Z_0).

Note that (2.23) allows to define, say, the nth coordinate as "time", i.e. $Z_t^n = t$: simply set $\mu_n(z) := 1$ and $\sigma_n^j(z) := 0$ for $j = 1,\dots,d$. This way, a deterministic dependency of the coefficients μ and σ on time can be incorporated in the above formulation (see example 2.23 below and the "fitting models" in section 3.4).

DEFINITION 2.20 (Consistent Parameter Process) *A* locally Consistent Parameter Process *for* $(G, \mathcal{Z})$ *is a diffusion process* $Z \in \Xi$ *with explosion time* $\tau > 0$*, such that for all* $Z_0 \in \mathcal{Z}$ *we have* $Z_{t\wedge\tau} \in \mathcal{Z}$ *and the family*

$$\hat{v}_t(x) := G(Z_{t\wedge\tau}; x) \qquad x \geq 0 \ ,$$

is a variance curve model.[9]

The pair (G, Z) *is then called* locally consistent. *It is called* globally consistent *if* $\tau = \infty$ *for all* $Z_0 \in \mathcal{Z}$. *We also say that* (G, Z) *are* strongly *consistent if* v *above is a strong variance curve model.*

Once we have determined a consistent pair (G, Z), an associated stock price is defined by choosing a correlation structure between v and the stock price process in form of a Brownian motion B (see theorem 2.8 and the subsequent remark 2.9). To preserve the Markov property of the joint process (Z, S), we impose some structure on the choice of B.

2.3.1 Markov Variance Curve Market Models

DEFINITION 2.21 *A* correlation function *is a measurable map* $\rho : \mathcal{Z} \times \mathbb{R}_{\geq 0} \to [-1, +1]^d$ *such that* $\|\rho(z, s)\|_2 = 1$ *for all* $(z, s) \in \mathcal{Z} \times \mathbb{R}_{\geq 0}$.[10]

DEFINITION 2.22 (Markov Variance Curve Market Model) *Assume* (G, Z) *is locally consistent with explosion time* $\tau > 0$ *and that* ρ *is a correlation function. Let the* ρ*-associated stock price* $S = (S_t)_{0\leq t\leq\tau}$ *be given as the unique solution to the equation*

$$\frac{dS_t}{S_t} = \sqrt{\zeta_t} \sum_{j=1}^{d} \rho^j(Z_t, S_t)\, dW_t^j \ , \quad \zeta_t := G(Z_t, 0) \ . \tag{2.24}$$

By definition, the process (S_t, Z_t) *is then Markovian and we call the triple* (G, Z, ρ) *a local* Markov variance curve market model *or* MVCMM.

It is called global *if* $\tau = \infty$.

Moreover, it is called strong *if* (G, Z) *is globally consistent and if the variance swap market model is strong, i.e. if all variance swaps and the stock price are true martingales.*

Proof that (2.24) admits a unique strong solution– Let $\tau > 0$ be the explosion time of Z and define the local martingales $M_t^j := \int_0^t \sqrt{G(Z_t, 0)}\, dW_t^j$ until τ. Let $u_t^j(x) := x\rho^j(Z_t; x)$. Equation (2.24) becomes

$$dS_t = \sum_{j=1}^{d} u_t^j(S_t)\, dM_t^j \qquad \{t \leq \tau\} \ .$$

Since $\|u_t(x) - u_t(y)\| \leq 2^d||x - y||$, existence and uniqueness up to τ follow from theorem 7 in Protter [P04] pg. 253. □

[9] Strictly speaking, the process Z depends on the starting point Z_0, hence we are actually speaking about a family of processes rather than a single process.

[10] The norm $|| \cdot ||_2$ is the usual L^2 norm, hence the condition above translates into $1 = \sum_{j=1,\dots,d}(\rho^j(z, s))^2$ for all (z, s).

Remark 2.23 (Local Volatility) *A "local volatility" such as Dupire's implied local volatility [D96] is also a Markov variance curve market model.*

We show a more general result: let $Z \in \Xi$, assume that ρ is a correlation function and that η is a suitable "local volatility" function such that

$$\frac{dS_t}{S_t} = \eta(S_t, Z_t) \sum_{j=1}^{d} \rho^j(S_t, Z_t)\, dW_t^j \tag{2.25}$$

has a unique strong strictly positive solution S which is a true martingale up to all finite T (note that as above, η can depend on time by imposing, for example, $Z_t^n = t$).

Let $Y = (Y^0, \dots, Y^m) := (S, Z^1, \dots, Z^m)$. This process uniquely solves

$$dY_t = \tilde{\mu}(Y_t)\, dt + \sum_{j=1}^{d} \tilde{\sigma}^j(Y_t)\, dW_t^j$$

with $\tilde{\mu}(z,s) = (0, \mu_1(z), \dots, \mu_m(z))$ and

$$\tilde{\sigma}(z,s) = \begin{pmatrix} s\,\eta(s,z)\rho^1(s,z) & \cdots & s\,\eta(s,z)\rho^d(s,z) \\ \sigma_1^1(z) & \cdots & \sigma_1^d(z) \\ \vdots & \ddots & \vdots \\ \sigma_m^1(z) & \cdots & \sigma_m^d(z) \end{pmatrix} .$$

By construction $Y \in \Xi$. The variance curve functional for (2.25) is given by

$$G(\tilde{z}; x) := \mathbb{E}\left[\, \eta(S_x, Z_x)^2 \,\middle|\, S_0 = \tilde{z}_0; Z_0 := (\tilde{z}_1, \dots, \tilde{z}_m) \,\right] .$$

Now, it is just a matter of notation to see that

$$\sqrt{G(Y_t; 0)} \sum_{j=1}^{d} \rho^j(S_t, Z_t)\, dW_t^j = S_t\, \eta(S_t, Z_t) \sum_{j=1}^{d} \rho^j(S_t, Z_t)\, dW_t^j = dS_t .$$

□

2.3.2 HJM-Conditions for Consistent Parameter Processes

We can now prove the following theorem, which is closely related to proposition 3.1.1 in [F01]. It paves the way to answer problem (**P2**) (also note theorem 5.20 in section 5.3.1 which provides a related result for "entropy swaps").

Theorem 2.24 (HJM-condition for Variance Curve Functionals) *A process $Z \in \Xi$ is locally consistent with $(G, \mathcal{Z})$ if and only if for each $Z_0 \in \mathcal{Z}$, the process Z stays in $\mathcal{Z}$ and*

$$\partial_x G(z; x) = \mu(z)\ \partial_z G(z; x) + \frac{1}{2}\sigma^2(z)\ \partial_{zz} G(z; x) \tag{2.26}$$

holds on $\mathcal{Z} \times \mathbb{R}_{\geq 0}$.

Moreover, Z is strongly consistent if and only if additionally $\tau = \infty$ and $G(Z_0; T) = \mathbb{E}\left[\, G(Z_T; 0) \,\right]$ holds for all finite T.

The above equation (2.26) is short-cut notation for

$$\begin{aligned}\partial_x G(z;x) &= \sum_{i=1}^{m} \mu_i(z)\,\partial_{z_i} G(z;x) \\ &+ \frac{1}{2}\sum_{i,k=1}^{m}\left(\sum_{j=1}^{d}\sigma_i^j(z)\sigma_k^j(z)\right)\partial_{z_i z_k} G(z;x)\ .\end{aligned}$$

Proof – First assume that Z is locally consistent with G. Then $\hat{v}_t(x) := G(Z_t;x)$ is a variance curve model and

$$d\hat{v}_t = \left(\mu\,\partial_z G + \frac{1}{2}\sigma^2\,\partial_{zz} G\right) dt + \sum_{j=1}^{d}\sigma^j\,\partial_z G\, dW_t^j$$

shows that

$$\partial_x G(Z_t;x) = \mu(Z_t)\ \partial_z G(Z_t;x) + \frac{1}{2}\sigma^2(Z_t)\ \partial_{zz} G(Z_t;x)$$

on $t \le \tau$ almost surely by theorem 2.13. In particular, this condition has to hold for each $Z_0 \in \mathcal{Z}$, which shows that indeed (2.26) must hold.

Now assume on the other hand that (2.26) holds. Using Itô and (2.26) it is clear that $v_t(T) := G(Z_t; T-t)$ is a local martingale up to τ for all T with

$$dv_t(T) = \sum_{j=1}^{d}\beta_t^j(T)\,dW_t^j\ ,\quad \beta_t^j(T) := \sum_{i=1}^{m}\sigma_i^j(Z_t)\,\partial_{z_i} G(Z_t;T-t) \in L_T^{\mathrm{loc}}\ . \tag{2.27}$$

This proves the theorem for the local case. The case of a strong variance curve is obvious. □

Theorem 2.24 gives us the required conditions for problem (**P2**) when a pair (G,Z) is consistent. However, it leaves the question open whether a process Z given by a pair (μ,σ) leaves $\mathcal{Z}$ at some stage or not. This will be treated in theorem 2.26 in the following section.

2.3.3 Extensions to Manifolds: When does Z stay in $\mathcal{Z}$?

In the following section, we want to discuss conditions on when Z stays inside the domain $\mathcal{Z}$ of $G(\cdot;x)$. Since the methods we want to employ require a notion of differentiability, we will now assume that $\mathcal{Z}$ is a regular sub-manifold with boundary. The reason why we include the case with boundary is that this situation arises in many examples, notably the "linearly mean-reverting" models (such as Heston's) which are discussed in chapter 3.

Definition 2.25 (Invariant manifold) *Let $(\mu,\sigma) \in \Xi$. A regular sub-manifold with boundary $\mathcal{Z} \subseteq \mathbb{R}^m_{\ge 0}$ is called* locally invariant *for Z if for any starting point $Z_0 \in \mathcal{Z}$, there exists a strictly positive stopping time η such that $Z_{t\wedge\tau} \in \mathcal{Z}$ for $t < \eta$.*

We call $\mathcal{Z}$ globally invariant *or just* invariant *if we can set $\eta = \infty$.*

Recall that if $\mathcal{Z}$ is a d-dimensional regular sub-manifold with boundary $\partial\mathcal{Z}$, then $\mathcal{T}_x\mathcal{Z}$ denotes the tangent space of $\mathcal{Z}$ in an interior point $x \in \mathcal{Z}$. By definition, the boundary $\partial\mathcal{Z}$ of $\mathcal{Z}$ is either empty or a $(d-1)$-dimensional manifold, and for a point $x \in \partial\mathcal{Z}$, its $(d-1)$-dimensional tangent space with respect to $\partial\mathcal{Z}$ is denoted by $\mathcal{T}_x\partial\mathcal{Z}$. For any regular sub-manifold, its closure is denoted by $\bar{\mathcal{Z}}$, and contains its boundary (which might be empty).

For any point $x \in \partial\mathcal{Z}$, we can find a smooth map $\varphi : U \to V$ from an open set U into a relatively open set $V \subset \mathbb{R}^{d-1} \times \mathbb{R}_{\geq 0}$, which generates the "inward pointing" tangent space $(\mathcal{T}_x\mathcal{Z})_{\geq 0}$ of $\mathcal{M}$ in x.[11] We will now follow Björk et al. [BC99] and derive a condition under which Z will stay in a regular manifold with boundary $\mathcal{Z}$.

Assume $\sigma \in C^1(\mathcal{Z})$, and define the vector

$$\Sigma(z) := \begin{pmatrix} \Sigma_1(z) \\ \vdots \\ \Sigma_m(z) \end{pmatrix} \tag{2.28}$$

with $\Sigma_i(z)$ given as the sum $\sum_{j=1}^{d} (\partial_z \sigma_i^j)\sigma^j(z)$, where we define

$$(\partial_z \sigma_i^j)\sigma^j(z) := \sum_{\ell=1}^{m} (\partial_{z_\ell} \sigma_i^j)(z)\sigma_\ell^j(z) \ . \tag{2.29}$$

THEOREM 2.26 *Assume that $\mathcal{Z} \in \mathbb{R}^m$ is a d-dimensional regular sub-manifold with boundary and let $Z \in \Xi$ with coefficients (μ, σ) and $\sigma \in C^1$.*

Then, $\mathcal{Z}$ is locally invariant for (μ, σ) if

$$\left.\begin{aligned} \mu(z) - \tfrac{1}{2}\,\Sigma(z) &\in \mathcal{T}_z\mathcal{Z} \\ \sigma^j(z) &\in \mathcal{T}_z\mathcal{Z} \end{aligned}\right\} \tag{2.30}$$

for all $z \in \mathrm{int}\mathcal{Z}$.

Moreover, if $\mathcal{Z}$ is closed in the relative topology, then it is globally invariant if additionally

$$\left.\begin{aligned} \mu(z) - \tfrac{1}{2}\,\Sigma(z) &\in (\mathcal{T}_z\mathcal{Z})_{\geq 0} \\ \sigma^j(z) &\in \mathcal{T}_z\partial\mathcal{Z} \end{aligned}\right\} \tag{2.31}$$

for all $z \in \partial\mathcal{Z}$.[12]

Proof – The proof is an application of Stratonovich-calculus. See also Filipovic/Teichmann [FT04] theorem 1.2 or Teichmann [T05] pg. 19 for a similar statement.

Step 1: We first look at the general case of a sub-manifold, i.e. assume $Z_0 \in \mathrm{int}\mathcal{Z}$. Then, there exists an open set U_0 (in the relative topology) such that $Z_0 \in U_0 \subset \mathcal{Z}$. The solution Y to a Stratonovich-SDE

$$dY_t = \eta(Y_t)\,dt + \varsigma(Y_t) \circ dW_t$$

starting at $Y_0 = Z_0$ will stay in $\mathcal{Z}$ until it leaves U_0, if and only if

$$\left.\begin{aligned} \eta(z) &\in \mathcal{T}_z\mathcal{Z} \\ \varsigma(z) &\in \mathcal{T}_z\mathcal{Z} \end{aligned}\right\}$$

for all $z \in U_0$: This follows since Stratonovich calculus obeys the same rules as the standard calculus.[13] Also, the exit time τ from U_0 is strictly positive, so $\mathcal{Z}$ is locally invariant for Y.

From that, it is also clear why the process may only stay *locally* in $\mathcal{Z}$: If the sub-manifold is open in the relative topology, Y can approach the boundary in a sequence of steps and so finally

[11] See Hirsch [H91] for an introduction into manifolds. Our notation follows Teichmann [T05].

[12] Note that $\partial\mathcal{Z}$ might be empty.

[13] For an introduction into this topic, see for example Roger/Williams [RW00] chapter V.

leave the manifold via its boundary (think of the circle around the open unit ball). Indeed, it can be shown that it can only leave the manifold at a boundary.

Step 2: Now consider the case where $\mathcal{Z}$ is closed in the relative topology, i.e. that $\partial\bar{\mathcal{Z}} \subset \mathcal{Z}$. By standard calculus, the condition $\eta(z) \in (\mathcal{T}_z\mathcal{Z})_{\geq 0}$ ensures that the pure drift term stays on the manifold. The second condition $\varsigma(z) \in \mathcal{T}_z\partial\mathcal{Z}$ on the other hand ensures that the diffusion term does not drive the solution out of the $(d-1)$-dimensional manifold $\partial\mathcal{Z}$, just as above, so the diffusion Y cannot leave $\mathcal{Z}$. This situation is illustrated in figure 2.1 below.

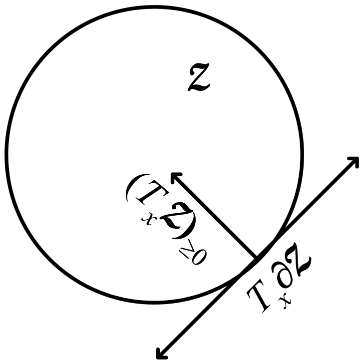


Figure 2.1: The tangent spaces $(\mathcal{T}_x\mathcal{Z})_{\geq 0}$ and $\mathcal{T}_x\partial\mathcal{Z}$ for a point $x \in \partial\mathcal{Z}$.

Step 3: The previous remarks can now be translated to our case using the transition between Stratonovich and Itô integral by way of the general formula

$$M^j \circ dW^j = M^j\, dW^j - \frac{1}{2} d\langle M^j, W^j\rangle$$

for some semi-martingale M.

Let $i = 1, \ldots, m$ and $j = 1, \ldots, d$. As before, σ_i^j is the volatility coefficient of Z^i with respect to the jth Brownian motion. We have

$$\begin{aligned} d\sigma_i^j(Z_t) &\overset{\text{Itô}}{=} \sum_{\ell=1}^m (\partial_{z_\ell}\sigma_i^j)(Z)\, dZ_t^\ell + (\cdots)\, dt \\ &= \sum_{\ell=1}^m (\partial_{z_\ell}\sigma_i^j)(Z) \sum_{k=1}^d \sigma_\ell^k(Z_t)\, dW_t^k + (\cdots)\, dt\ , \end{aligned}$$

and consequently

$$\begin{aligned} d\langle\sigma_i^j(Z), W^j\rangle_t &= \sum_{\ell=1}^m d\langle \int_0^{\cdot} (\partial_{z_\ell}\sigma_i^j)(Z_s) \sum_{k=1}^d \sigma_\ell^k(Z_s) dW_s^k\ ,\ W^j\rangle_t \\ &= \sum_{\ell=1}^m \Big((\partial_{z_\ell}\sigma_i^j)(Z_t)\sigma_\ell^j(Z_t) \Big)\ dt \\ &\overset{(2.29)}{=}: (\partial_z\sigma_i^j)\sigma^j\, dt\ . \end{aligned}$$

By defining Σ as in (2.28) we get

$$dZ_t = \left(\mu(Z_t) - \frac{1}{2}\Sigma(Z_t)\right)\ dt + \sum_{j=1}^d \sigma^j(Z_t) \circ\ dW_t^j\ ,$$

to which step 1 and 2 of the proof apply. □

We are hence able to answer (**P2**) satisfactory in the case where $\mathcal{Z}$ is a sub-manifold:

CONCLUSION 2.27 (Solution to problem (**P2**)) *To check whether Z and G are consistent, we apply theorem 2.24 to find out whether the process is consistent on $\mathcal{Z}$, and theorem 2.26 to determine whether it also stays (at least locally) in $\mathcal{Z}$.*

2.4 Variance Curve Models in Hilbert Spaces

We will now focus on problem (**P3**): Given $\hat{v}$ now as a solution of a general SDE of the form (2.18), and a curve functional G such that $G(\mathcal{Z})$ is a sub-manifold of a Hilbert-space $\mathcal{H}$, under which conditions on the coefficients of $\hat{v}$ can we find a consistent parameter process Z such that

$$\hat{v}_t = G(Z_t) \ ?$$

(We now drop the x-argument since we understand $\hat{v}_t$ and $G(Z_t)$ in this section as elements of a function space.) Note that such a representation is also an efficient way to ensure non-negativity of the process $\hat{v}$.

To be able to approach this question, we have to impose some regularity on the possible curves of $\hat{v}$. Indeed, we will employ the theory of stochastic differential equations in Hilbert spaces, the standard reference on which is da Prato/Zabcyk [PZ92]; also see Teichmann [T05]. We will closely follow Björk/Svensson [BS01], Filipovic/Teichmann [FT04] and Teichmann [T05].

We remain on the space $\mathbb{W} = (\Omega, \mathcal{F}_\infty, \mathbb{P}, \mathbb{F})$ which supports an extremal d-dimensional Brownian motion W. Additionally we assume that we are also given a Hilbert-space $\mathcal{H}$, which will contain our forward variance curves.[14]

In $\mathcal{H}$, we assume $\hat{v}$ is given as a solution to a stochastic differential equation of the type

$$d\hat{v}_t = \partial_x \hat{v}_t \, dt + \sum_{j=1}^{d} b^j(\hat{v}_t) \, dW_t^j \tag{2.32}$$

with locally Lipschitz vector fields $\beta^1, \ldots, \beta^d : U \subset \mathcal{H} \to \mathcal{H}$ where U is an open set. A (mild) solution[15] of such an equation typically only exists up to a strictly positive stopping explosion time η, hence we focus on questions of local consistency. The operator $\partial_x : \mathrm{dom}(\partial_x) \subset \mathcal{H} \to \mathcal{H}$ is the generator of the strongly continuous semigroup $(T_t)_{t\geq 0}$ of shift operators $(T_t\hat{v})(x) := \hat{v}(x+t)$; see da Prato/Zabcyk [PZ92] for details.

ASSUMPTION 4 *The set $\mathcal{G} := G(\mathcal{Z}) \subset \mathcal{H}$ is a sub-manifold with boundary $\partial\mathcal{G}$.*[16]

DEFINITION 2.28 (Locally Consistency and FDR) *We say $\hat{v} = (\hat{v}_t)_{0\leq t\leq \eta}$ is* locally consistent *with G if there exist a locally consistent $(\mu, \sigma) \in \Xi$ for G with explosion time $\tau > 0$ such that*

$$\hat{v}_t = G(Z_t)$$

[14] For examples of suitable Hilbert-spaces, see Filipovic [F01].

[15] For concepts of solutions for equations in Hilbert-spaces, see da Prato/Zabcyk [PZ92] or Teichmann [T05].

[16] The boundary is finite-dimensional by construction.

for all $t \leq \eta \wedge \tau$. We call the pair (G, Z) a finite dimensional representation *or* FDR *of the variance curve $\hat{v}$.*

Let us define the Stratonovic drift of $\hat{v}$,

$$b^0(\hat{v}) := \partial_x \hat{v} - \frac{1}{2}\sum_{j=1}^{d}(Db^j)(\hat{v})\; b^j(\hat{v})$$

where $(Db^j)(\hat{v})$ denotes the Frechet-derivative of b^j along $\hat{v}$. Note that the drift b^0 is only well-defined for $\hat{v} \in \mathrm{dom}(\partial_x)$.

Theorem 2.29 *The process $\hat{v} = (\hat{v}_t)_{0 \leq t \leq \eta}$ with $\hat{v}_0 \in \mathcal{G}$ is locally consistent with G if and only if*

(a) $\mathcal{G} \subset dom(\partial_x)$.

(b) For all $\hat{v} \in \mathcal{G} \setminus \partial\mathcal{G}$ and for $j = 0, \ldots, d$,

$$b^j(\hat{v}) \in T_{\hat{v}}\mathcal{G} \; . \tag{2.33}$$

(the tangent space $T_{\hat{v}}\mathcal{G}$ in $\hat{v} = G(z)$ is given by Img $\partial_z G(z)$*).*

(c) For all $\hat{v} \in \partial\mathcal{G}$,

$$b^0(\hat{v}) \in (T_{\hat{v}}\mathcal{G})_{\geq 0} \quad \textit{and} \quad b^j(\hat{v}) \in T_{\hat{v}}\partial\mathcal{G} \tag{2.34}$$

for $j = 1, \ldots, d$.[17]

For a proof, see Filipovic/Teichmann [FT04] or theorem 13 in [T05]. We also obtain:

Corollary 2.30 *If $\hat{v}$ is locally consistent with G and if G is invertible on $\mathcal{Z}$, then the parameter process (μ, σ) specified by*

$$\sigma^j(z) = (\partial_z G)(z)^{-1}\; b^j(G(z))$$

for $j = 1, \ldots, d$ and

$$\mu(z) := (\partial_z G)(z)^{-1}\; b^0(G(z)) + \sum_{j=1}^{d}(\partial_z \sigma^j)(z)\,\sigma^j(z) \; .$$

is in Ξ and locally consistent with G.

Conclusion 2.31 (Local solution to problem (**P3**)) *At least locally, Theorem 2.29 solves problem (**P3**): a variance curve model admits an FDR (G, Z) if and only if (2.33) and (2.33) are satisfied. If G is invertible, the parameter process Z is given in Corollary 2.30.*

[17] We used $(T_{\hat{v}}\mathcal{G})_{\geq 0}$ to denote the inward pointing tangent-space of $\mathcal{G}$ in the boundary point $\hat{v}$.

Chapter 3

Examples

In this chapter, we will apply the theory developed in the previous chapter to some examples of variance curve models. We will mainly focus on exponential-polynomial variance curves, but also discuss a few other approaches.

The main purpose of this section is to show how theorem 2.24 restricts the possible choices of parameter processes for a given functional G. According to theorem 2.24, the coefficients (μ, σ) of every consistent process $Z \in \Xi$ must satisfy (2.26), i.e.

$$\begin{aligned} \partial_x G(z;x) &= \sum_{i=1}^{m} \mu_i(z)\, \partial_{z_i} G(z;x) \\ &+ \frac{1}{2} \sum_{i,k=1}^{m} \left(\sum_{j=1}^{d} \sigma_i^j(z) \sigma_k^j(z) \right) \partial_{z_i z_k} G(z;x) \; . \end{aligned} \tag{3.1}$$

Hence, if G is given, we need to find $(\mu, \sigma) \in \Xi$ such that (3.1) is satisfied.

3.1 Exponential-Polynomial Variance Curve Models

Definition 3.1 *The family of* Exponential-Polynomial Curve Functional *is parameterized by* $z = (z_1, \dots, z_r; z_{r+1}, \dots, z_m) \in {\mathbb{R}_{>0}}^r \times \mathbb{R}^{m-r}$ *and given as*

$$G(z;x) = \sum_{i=1}^{r} p_i(z;x) e^{-z_i x} \tag{3.2}$$

where p_i are polynomials of the form $p_i(z,x) = \sum_{j=0}^{N} a_{ij}(z) x^j$ with coefficients a_{ij} such that $p_i \geq 0$ λ^d-as. W.l.g. we can assume that $\deg(p^i) > \deg(p^{i+1})$.[1]

(Also compare Björk/Svensson [BS01] and Filipovic [F01].)

We assume that any parameter process has $Z^i \neq Z^j$ for $i \neq j$, since otherwise we can just rewrite (3.2) accordingly. Also note that $\int_0^T G(z;x)\, dx < \infty$ for all $T < \infty$.

Lemma 3.2 *Let $Z \in \Xi$ be a parameter process consistent with an exponential-polynomial variance curve functional.*

Then, the first r coordinates $Z^1, \dots, Z^r$ are constant.

[1] We denote by $\deg(p)$ the degree of a polynomial p.

Proof – We have

$$\partial_x G(z,x) = -z_i \sum_{i=1}^{r} p_i(z;x)e^{-z_i x} + \sum_{i=1}^{r} \partial_x p_i(z;x)e^{-z_i x} \tag{3.3}$$

$$\partial_{z_k} G(z,x) = -p_k(z;x)xe^{-z_k x}1_{k\le r} + \sum_{i=1}^{r} \partial_{z_k} p_i(z;x)e^{-z_i x} \tag{3.4}$$

$$\partial^2_{z_k z_k} G(z,x) = p_k(z;x)x^2 e^{-z_k x}1_{k\le r} - \partial_{z_k} p_k(z;x)xe^{-z_k x}1_{k\le r} \tag{3.5}$$

$$+ \sum_{i=1}^{r} \partial^2_{z_k z_k} p_i(z;x)e^{-z_i x} \tag{3.6}$$

The terms $\partial^2_{z_k z_k} G(z,x)$ with $k \le r$ are the only terms in (3.1) which involve polynomials of degree $\deg(p_i) + 2$ as factors in front of the exponentials $e^{-z_i x}$. Since we choose the z_i distinct, and because neither μ nor σ depends on x, this implies that

$$0 = \sum_{j=1}^{d} \sigma^j_k(z)\sigma^j_k(z)$$

for $k \le r$, which implies that the vector σ_k must vanish. In other words, the states z_k for $k \le r$ cannot be random.

Next, we use (3.4) and find with the same reasoning (now applied to the polynomials of degree $\deg(p_i) + 1$) that $\mu_i = 0$ for $i \le r$, so Z^i must be a constant. □

We will now present two particular exponential-polynomial curve functionals. In the light of lemma 3.2, we will keep the exponentials constant but investigate the possible dynamics of the remaining parameters.

Example 3.3 (Linearly Mean-Reverting Variance Curve Models) *The Functional*

$$G(z;x) := z_2 + (z_1 - z_2)e^{-\kappa x} \ .$$

with $z \in \mathcal{Z} := \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0}$ *is consistent with* $Z \in \Xi$ *if* $\mu_1(z) = \kappa(z_2 - z_1)$ *and* $\mu_2(z) = 0$ *(that is,* Z^2 *must be a martingale). The volatility parameters can be freely specified, as long as* Z^1 *and* Z^2 *remain non-negative.*

We call such a model a linearly mean-reverting variance curve model.

Proof – Theorem 2.24 with equation (2.26) implies that we have to match

$$-\kappa(z_1 - z_2)e^{-\kappa x} = \mu_1(z)e^{-\kappa x} + \mu_2(z)(1 - e^{-\kappa x}) \ .$$

Since the left hand side has no term constant in x, we must have $\mu_2(z) = 0$ (i.e. Z^2 is a martingale), and then $\mu_1(z) = \kappa(z_2 - z_1)$. □

A popular parametrization is $\sigma_2 = 0$ and $\sigma_1(z_1) = \nu\sqrt{z_1}$ for some $\nu > 0$, which has been introduced by Heston [H93]: Z^1 is then the square of the short-volatility of the associated stock price process. Another possible choice for the parameters in example 3.3 is

$$\mu(z) = \begin{pmatrix} \kappa(z_2 - z_1) \\ 0 \end{pmatrix} \qquad \sigma(z) = \begin{pmatrix} \nu z_1^\alpha & 0 \\ \eta\rho z_2 & \eta\hat{\rho} z_2 \end{pmatrix}$$

with constants $\alpha \in [\frac{1}{2}, 2]$, $\nu, \eta \in \mathbb{R}_{>0}$, $\rho \in (-1, 0]$ and $\hat\rho = \sqrt{1-\rho^2}$. In this example, the mean-reversion level z_2 is a geometric Brownian motion (with the intuitive drawback that it can become very large).

EXAMPLE 3.4 (Fitting Heston to the market) *In the light of the discussion in Section 2.2.3, let us show an approach here to fit a Heston-model to an observed variance swap curve while retaining computational tractability.*

To this end, consider Heston's model [H93] with a time-dependent mean-reversion level,

$$\begin{aligned} dZ_t^1 &= \kappa(\theta(Z_t^2) - Z_t^1)\,dt + \nu\sqrt{Z_t^1}\,dW_t^1 \\ dZ_t^2 &= dt \end{aligned}$$

with the associated stock price process given by a constant correlation ρ. (Note that $Z_t^2 = Z_0^2 + t$.)

Assume now that we observe a market variance curve $\hat u_0 \in C^1[\mathbb{R}_{\geq 0}]$ and let $\theta(x) := \kappa\hat u_0(x) + \partial_x \hat u_0(x)$. If $\theta(x) \geq 0$ (such that $Z_t^1 \geq 0$), then Z is fits the market in the sense that

$$\mathbb{E}\left[\, Z_x^1 \,\middle|\, Z_0 = (u_0(0), 0)\,\right] = \hat u_0(x)\ .$$

The characteristic function of the logarithm of the stock price in this model can be computed using standard methods; see Bermudez et al. [BBFJLO06] for details.

The next functional is a generalization of the linearly mean-reverting case above. It is akin to Svensson's model for interest rate forward curves.

EXAMPLE 3.5 (Double Mean-Reverting Variance Curve Models) *Let $c, \kappa > 0$ constant and let $z = (z_1, z_2, z_3) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2_{>0}$. The Curve Functional*

$$G(z; x) := z_3 + (z_1 - z_3)e^{-\kappa x} + (z_2 - z_3)\begin{cases} \frac{\kappa}{\kappa - c}\left(e^{-cx} - e^{-\kappa x}\right) & (\kappa \neq c) \\ \kappa x\, e^{-\kappa x} & (\kappa = c) \end{cases} \tag{3.7}$$

is consistent with any parameter process (μ, σ) such that

$$\begin{aligned} dZ_t^1 &= \kappa(Z_t^2 - Z_t^1)\,dt + \sigma_1(Z_t)\,dW_t \\ dZ_t^2 &= c(Z_t^3 - Z_t^2)\,dt + \sigma_2(Z_t)\,dW_t \\ dZ_t^3 &= \sigma_3(Z_t)\,dW_t \end{aligned}$$

and is called a double mean-reverting variance curve model*.*

Proof – First let $\kappa = c$. Then,

$$\partial_x G(z, x) = \{-\kappa(z_1 - z_3 + \kappa x(z_2 - z_3)) + \kappa(z_2 - z_3)\}\ e^{-\kappa x}$$

and

$$\begin{aligned} \partial_{z_1} G(z, x) &= e^{-\kappa x} \\ \partial_{z_2} G(z, x) &= \kappa x e^{-\kappa x} \\ \partial_{z_3} G(z, x) &= 1 - e^{-\kappa x} + \kappa x e^{-\kappa x}\ . \end{aligned}$$

This yields $\mu_3 = 0$. The remaining equation reads

$$-\kappa(z_1 - z_2) - \kappa^2 x(z_2 - z_3) = \mu_1(z) + \kappa x \mu_2(z)$$

such that $\mu^2(z) = \kappa(z_3 - z_2)$ and $\mu^1(z) = \kappa(z_2 - z_1)$.

Now assume $\kappa \neq c$. Let $\gamma := \frac{\kappa}{\kappa - c}$. Then,

$$G(z; x) := z_3 + (z_1 - z_3 - \gamma(z_2 - z_3))e^{-\kappa x} + \gamma(z_2 - z_3)e^{-cx}$$

and therefore

$$\begin{aligned}
\partial_x G(z, x) &= -\kappa \left\{ z_1 - z_3 - \gamma(z_2 - z_3) \right\} e^{-\kappa x} - c(z_2 - z_3)\gamma e^{-cx} \\
\partial_{z_1} G(z, x) &= e^{-\kappa x} \\
\partial_{z_2} G(z, x) &= -\gamma e^{-\kappa x} + \gamma e^{-cx} \\
\partial_{z_3} G(z, x) &= 1 + (\gamma - 1)e^{-\kappa x} - \gamma e^{-cx} \; .
\end{aligned}$$

Hence, once more $\mu_3 = 0$. Furthermore $\mu_2 = c(z_3 - z_2)$ and finally $\mu_1 = \kappa(z_3 - z_1)$. □

This turns out to be a flexible and applicable variance curve functional. Figure 3.1 shows a few typical shapes of the variance swap strike term structure, i.e. of the curve $T \mapsto \sqrt{\int_0^T G(z; x)\, dx / T}$ (this is the market standard to quote a variance swap). See also figure 6.3 on page 86 which illustrates the impact of changing the parameters on the shape of the curve.

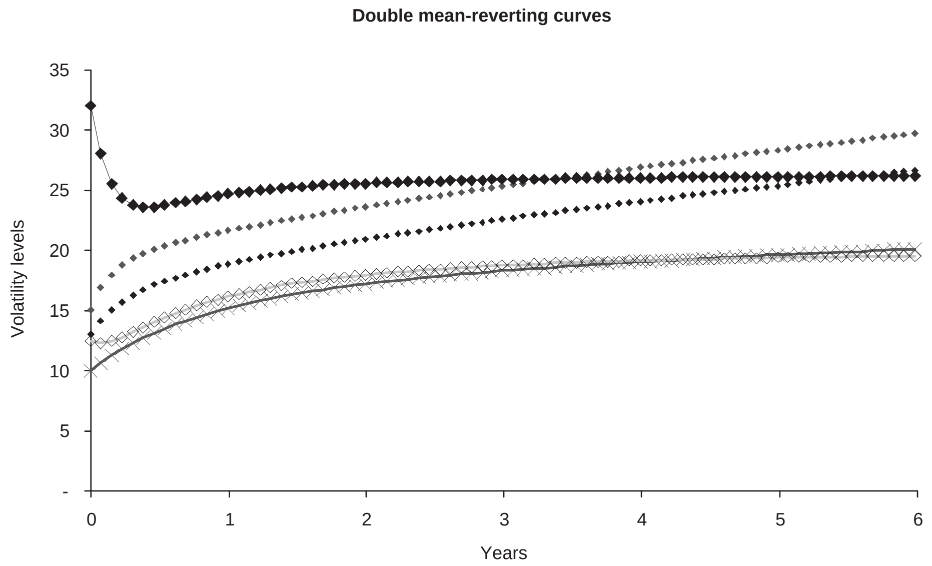


Figure 3.1: Various shapes of the variance swap prices given for various parameterizations of the functional (3.7). The graph shows "variance swap volatilites" $\sqrt{\int_0^T G(z; x)\, dx / T}$, cf. (1.2).

At the time of writing, the variance functional (3.7) fits the variance swap market of major indices well, so this kind of double mean-reverting model is a good candidate for a variance curve model (a similar model has been proposed by Duffie et al. [DPS00] who use $\sigma_1(z) = \sqrt{z_1}$ and $\sigma_2(z) = \sqrt{z_2}$ with a particular sparse correlation structure).

We will discuss the implementation of a three-model with all technical details in chapter 6.

Example 3.6 *The one-factor model*

$$\begin{aligned}
d\zeta_t &= \kappa(\theta(t) - \zeta_t)\, dt + \nu \sqrt{\zeta_t}\, dW_t \\
d\theta(t) &= c(m - \theta(t))\, dt
\end{aligned}$$

is also consistent with the variance swap curve functional (3.7). In such a "Heston model with time-dependent mean-reversion speed", European options written on the stock can still be evaluated relatively efficient using Fourier-inversion (there is no need to solve a Ricatti equation). See Bermudez et al. [BBFJLO06] for details.

3.2 Exponential Curves

As in (3.2), let $(p_i)_{i=1,\dots,r}$ be polynomials and let

$$g(z;x) = \sum_{i=1}^{r} p_i(z;x) e^{-z_i x}$$

with $z = (z_1, \dots, z_r; z_{r+1}, \dots, z_m) \in {\mathbb{R}_{>0}}^r \times \mathbb{R}^{m-r}$. Set

$$G(z;x) := \exp(g(z;x)) \ . \tag{3.8}$$

Using theorem 2.24, a necessary condition for a consistent pair is

$$\partial_x g(z;x) = \mu(z)\, \partial_z g(z;x) + \frac{1}{2}\sigma^2(z) \left\{ (\partial_z g(z;x))^2 + \partial_{zz} g(z;x) \right\} \ , \tag{3.9}$$

where $(\partial_z g(z;x))^2 = \sum_{i,j=1}^m \partial_{z_i} g(z;x) \partial_{z_j} g(z;x)$. As a result, we obtain the following lemma,[2] whose proof is omitted because it is very similar to the proof of lemma 3.2.

LEMMA 3.7 *If Z is a consistent parameter process for G, then its coordinates $Z^1, \dots, Z^m$ are constant. Moreover, there must be at least one pair $i \neq j$ such that $Z^i = 2Z^j$, otherwise Z is entirely constant.*

As an immediate consequence, we have

EXAMPLE 3.8 (Exponential Mean-Reverting Models) *Let*

$$g(z;x) = z_2 + (z_1 - z_2) e^{-\kappa x} + \frac{z_3}{4\kappa}(1 - e^{-2\kappa x})$$

with $(z_1, z_2, z_3) \in \mathcal{Z} := \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{>0}$.

Then, $\mu_1(z) = \kappa(z_2 - z_1)$, $\sigma_1(z) = \sqrt{z_3}$ and $\mu_2 = \mu_3 = \sigma_2 = \sigma_3 = 0$, i.e. the only consistent factor model is the exponential Ornstein-Uhlenbeck stochastic volatility model discussed in depth by Fouque et al. in [FPS00].

Proof – The result is a relatively straight-forward consequence of (2.26) given that μ and σ must be defined for all $z \in \mathcal{Z}$ and that $\sigma_2 \geq 0$. □

3.3 Variance Swap Volatility Curves

In this section, we want to discuss a few variance curve term-structure interpolation schemes in terms of a *volatility* function, i.e. schemes where the variance swap price is given as a functional

$$\hat{V}_t(x) := x\, \Sigma^2(Z_t; x) \quad \text{where} \quad \Sigma(z;x) = z_1 + z_2 w(x) \tag{3.10}$$

[2] Compare theorems 3.6.1 and 3.6.2 from Filipovic [F01] pg.52ff.

for some volatility "term-structure" function w.[3]

Such curves arise if standard forms of *implied volatility* term-structure functionals are employed to model the variance swap curve. This approach might be appealing if the implied volatility term-structure is well captured by a particular choice of w.[4] A few choices which have been considered for implied volatility interpolation are

$$w_0(x) := 0 \tag{3.11}$$
$$w_1(x) := \ln(1+x) \tag{3.12}$$
$$w_2(x) := \sqrt{\epsilon + x} \tag{3.13}$$
$$w_3(x) := 1/\sqrt{\epsilon + x} \tag{3.14}$$

The first case w_0 is called the "Black&Scholes" case, since the term structure of variance swaps is linear. See also Haffner pg. 87 in [H04] for a discussion of implied volatility term-structure interpolation.

In our previous notation,

$$G(z;x) := \partial_x \left(\Sigma(z;x)^2 x\right) = \Sigma(z;x)^2 + 2\Sigma(z;x)\partial_x\Sigma(z;x)x$$

i.e.

$$\begin{aligned} G(z;x) &= (z_1^2 + 2z_1z_2w(x) + z_2^2w^2(x)) + 2(z_1 + z_2w(x))z_2x\partial_x w(x) \\ &= z_1^2 + 2z_1z_2(w(x) + \partial_x w(x)x) + z_2^2(w^2(x) + 2w(x)x\partial_x w(x)) \ . \end{aligned}$$

We have to show that

$$\partial_x G(z;x) = \mu(z)\,\partial_z G(z;x) + \frac{1}{2}\,\sigma^2(z)\,\partial_{zz}G(z;x) \tag{3.15}$$

holds.

Corollary 3.9

(a) Case (3.11) is consistent iff $\mu_1(z_1) = 0$, i.e. Z^1 is a non-negative martingale.

(b) Case (3.11) is the only case of (3.11)-(3.14) which is consistent.

Proof – The first statement is obvious. We prove the second statement for w_1, since it works similarly for w_2 and w_3. Hence let $w(x) = w_1(x) = \ln(1+x)$. For the remainder of section we denote $w'(x) := \partial_x w(x)$.

Focusing on (3.15), we obtain

$$\begin{aligned} \partial_x G(z;x) &= 2z_1z_2\left\{2w'(x) + w''(x)x\right\} \\ &\quad +2z_2^2\left\{2w'(x)w(x) + w'(x)^2x + w''(x)w(x)x\right\} \\ \partial_{z_1}G(z;x) &= 2z_1 + 2z_2\left\{w(x) + w'(x)x\right\} \\ \partial_{z_2}G(z;x) &= 2z_1\left\{w(x) + w'(x)x\right\} + 2z_2\left\{w^2(x) + 2w(x)w'(x)x\right\} \\ \partial_{z_1z_1}G(z;x) &= 2 \\ \partial_{z_2z_2}G(z;x) &= 2\left\{w^2(x) + 2w(x)w'(x)x\right\} \\ \partial_{z_1z_2}G(z;x) &= 2\left\{w(x) + w'(x)x\right\} \ . \end{aligned}$$

[3]In practise, variance swaps are quoted in terms of their volatility.

[4]Gatheral [Ga04] notes that in some stock price models, the shape of the term-structure of implied volatility is similar to the shape of the variance swap curve

We now show for w_1 that $z_2 = 0$ and $\mu_1 = 0$, i.e. that the functional degenerates to the Black & Scholes case. First, we compute the derivatives

$$w_1(x) = \ln(1+x) \ , \quad w_1'(x) = \frac{1}{1+x} \quad \text{and} \quad w_1''(x) = -\frac{1}{(1+x)^2} \ .$$

We therefore have

$$\partial_x G(z;x) = 2z_1z_2 \left\{ \frac{2}{1+x} - \frac{x}{(1+x)^2} \right\} + 2z_2^2 \left\{ 2\,\frac{\ln(1+x)}{1+x} + \frac{x - x\ln(1+x)}{(1+x)^2} \right\} \ .$$

Now note that none of the terms $\partial_{z_i} G(z;x)$ or $\partial^2_{z_i z_j} G(z;x)$ contains terms in $1/(1+x)^2$. Hence, to satisfy (3.15), for all $x \geq 0$, we must have

$$0 = -2z_1z_2x + 2z_2^2(x - x\ln(1+x)) \ . \tag{3.16}$$

Assume $z_2 \neq 0$. Then, (3.16) implies $z_1 = z_2(1 - \ln(1+x))$, which is not possible. Hence $z_2 = 0$ and the curve G reduces to the "Black & Scholes case" w_0. □

Corollary 3.9 has also shown that the variance curve functional $G(z,x) := z_1$ yields a consistent variance curve model for the parameter process Z if $\mu_1 = 0$.

A particular example is a model closely linked to the *SABR* model [HKLW02] with $\beta = 1$,

$$\left.\begin{array}{rcl} d\zeta_t &=& \sigma\,\zeta_t\,dW_t^1 \\ dX_t &=& \sqrt{\zeta_t}\,d(\rho W_t^1 + \sqrt{1-\rho^2}W_t^2) \\ S_t &=& \mathcal{E}_t(X) \end{array}\right\} \tag{3.17}$$

with log-normal short variance. Note that this model can be turned into a fitting model by adding a time-dependent drift to the geometric Brownian motion ζ. We generalize this idea in the following section.

3.4 Fitting Models

In this section, we show two techniques how variance curve models can fit an initial term structure of variance swaps perfectly. In both cases assume that at time 0, we observe an implied variance swap price curve

$$V_0(T) = \int_0^T v_0(x)\,dx \ . \tag{3.18}$$

with a differentiable forward variance curve $v_0 > 0$. We say a consistent variance curve model (G,Z) *fits the market* if

$$G(Z_0;x) = v_0(x) \ .$$

Multiplicative Fitting

We start with an example of a "fitting" log-normal type model (which is essentially Dupire's model [Du04]; cf. also remark 2.17) to motivate this approach:

EXAMPLE 3.10 (Dupire's [Du04] fitting log-normal short variance model) *Assume ν is a non-negative continuous function. Let*

$$G(z_1, z_2; x) := v_0(x + z_2)z_1 \ .$$

Then, the parameter process Z with

$$\mu(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad and \qquad \sigma(z) = \begin{pmatrix} \nu(Z_t^2)\, Z_t^1 \\ 0 \end{pmatrix} \tag{3.19}$$

(such that $Z_t^2 = Z_0^2 + t$) is consistent with G and the short variance ζ is given as

$$\zeta_t = v_0(t)\, \mathcal{E}_t \left(\int_0^\cdot \nu(u)\, dW_u^1 \right) = v_0(t) Z_t^1 \ . \tag{3.20}$$

In particular, the model fits the market: $G(1, 0; T) = v_0(T)$ for all T.

REMARK 3.11 *For ν constant, the classic extended Black&Scholes model is recovered.*

Such a model has also been discussed by Bergomi [B05], who's model is essentially the same approach based on a two-factor mean-reverting log-normal model. In the context of these models, note also Jourdain's article [J04] where the loss of the martingale property for such models is discussed if the correlation structure is misspecified. For example, if the correlation between W^1 above and the Brownian motion B which drives the stock (cf. (2.14)) is positive, then the associated stock price process to (3.19) is only a local martingale. We comment on pricing and hedging with local martingales in section 4.2.2.

Proof – It is evident that $Z_t^2 = t$ and therefore also that $Z_t^1 = \mathcal{E}_t \left(\int_0^\cdot \nu(u)\, dW_u^1 \right)$. As for (3.1), observe that

$$\begin{aligned} \partial_x G(z, x) &= \partial_x v_0(x + z_2) z_1 \\ \partial_{z_1} G(z, x) &= v_0(x + z_2) \\ \partial_{z_2} G(z, x) &= \partial_x v_0(x + z_2) z_1 \ , \end{aligned}$$

which shows already that (G, Z) are consistent. □

The same approach can be applied for any variance curve models: if we take an arbitrary given variance curve model, we can always scale it by the ratio of the market forward variance curve and the model's own forward variance curve. The result is a model which perfectly fits the market:

EXAMPLE 3.12 (Multiplicative Fitting) *Let $(\tilde{G}, \tilde{Z})$ with $(\tilde{\mu}, \tilde{\sigma})$ be consistent and let $Z_0 \in \mathcal{Z} \subset \mathbb{R}^m$ such that $\tilde{G}(\tilde{Z}_0; \cdot) > 0$. Assume as before we observe v_0 as in (3.18). Then let $z \in \mathbb{R}^{m+1}$ and define*

$$G(z; x) := G_0(x + z_{m+1})\, \tilde{G}(z_1, \dots, z_m; x)$$

with

$$G_0(x) := \frac{v_0(x)}{\tilde{G}(Z_0; x)} \ .$$

Then, G is consistent with $Z = (Z_t)_t$ where $Z_t = (\tilde{Z}_t^1, \dots, \tilde{Z}_t^m, t)$ and fits the market. The short variance is given as

$$\zeta_t = G_0(t)\, \tilde{G}(Z_t; 0) \ .$$

Proof – Equation (3.1) is satisfied and Z is in Ξ by construction. □

Fitting Mean-Reverting Models

The above approach can obviously also be applied to the mean-reverting type models of section 3.1, for example

$$d\tilde{Z}_t^1 = \kappa(\theta - \tilde{Z}_t^1)\,dt + \sigma(\tilde{Z}_t^1)\,dW_t^1 \tag{3.21}$$

and $\tilde{G}(z,x) = \theta + (z_1 - \theta)e^{-\kappa x}$. The advantage there is that for these models, the volatility coefficient σ can be arbitrary (as long as it ensures that the process remains positive) without altering the shape of the variance curve of the model $\tilde{G}$. For example, we can choose

$$\sigma(z) = 1_{z\le K_1}\,z^\alpha + 1_{z>K_1}\,K^\alpha\frac{K_2 - z\wedge K_2}{K_2 - K_1}$$

for $\alpha \in (1/2, 1]$ and two large but finite constants $\theta < K_1 < K_2$. In this case, (3.21) still has a non-explosive unique solution (because σ is bounded) and this solution is bounded. Hence, the associated stock price is a martingale for which all moments exists. The difference between using a linearly mean-revering model and a log-normal model as above is that if we want to impose a boundary on the log-normal model (3.20), it will change the variance curve function $\tilde{G}$ of the model in a way which is numerically much more expensive.[5]

However, "multiplicative fitting" of a model may not be the most natural approach. Indeed, in the case of linearly mean-reverting models, it seems to be more natural to modify the level of mean-reversion to fit the market curve:

EXAMPLE 3.13 (Fitting mean-reverting model) *Fix $\kappa > 0$, $\alpha \in [\frac{1}{2}, 1]$, $\nu > 0$ and let θ be a strictly positive differentiable function.*

$$G(z_1, z_2; x) \;:=\; z_1 e^{-\kappa x} + \kappa\int_0^x e^{-\kappa(x-s)}\theta(s + z_2)\,ds$$

has a consistent parameter process with

$$\mu(z) = \begin{pmatrix} \kappa(\theta(z_2) - z_1) \\ 1 \end{pmatrix} \quad \textit{and} \quad \sigma(z) = \begin{pmatrix} \nu\, z_1^\alpha \\ 0 \end{pmatrix}.$$

Once again, $Z_t^2 = t$.

Moreover, if

$$\theta(x) := v_0(x) + \frac{1}{\kappa}\partial_x v_0(x) \tag{3.22}$$

is non-negative, then the model fits the market: $G(v_0(0), 0; T) = v_0(T)$.

[5] Too see this, assume that we have the exponential mean-reverting model of example 3.8 in the form $d\tilde{Z}_t^1 = -\kappa\tilde{Z}_t^1\,dt + \nu dW_t^1$ with $\tilde{G}(z_1; x) := e^{z_1 e^{-\kappa t} + \frac{1}{2}\frac{1-e^{-2\kappa t}}{2\kappa}}$. In this model, the short variance is given as $\zeta_t = e^{Z_t^1}$ which can explode under the stock price measure: indeed, the stock becomes a strict local martingale if $d\langle W^1, B\rangle_t = \rho\,dt$ for a positive correlation ρ (cf. Jourdain [J04]). To alleviate this problem, we could try to bound the short variance by some constant K, say $\hat{\zeta}_t := \zeta_t \wedge K$. But this, in turn, will alter the variance curve function $\hat{G}(z_1; x) := \mathbb{E}[\,\zeta_x \wedge K \mid \zeta_0 = z_1\,]$ in a non-linear way – essentially,since $\zeta_x \wedge K = \zeta_x - (\zeta_x - K)^+$, we see that the function G is essentially $\tilde{G}$ less the Black&Scholes call price function.

The short variance in this model is $\zeta_t = G(Z_t; 0) = Z_t^1$ as in the mean-reverting models we saw before. Also note that the non-negativity condition on (3.22) essentially means that the market must have the form $v_0(x) = e^{-\kappa x} u(x)$ for a non-decreasing function u.

Proof – For (3.1), observe that

$$\begin{aligned}
\partial_x G(z,x) &= -\kappa z_1 e^{-\kappa x} + \kappa\theta(x+z_2) - \kappa^2 \int_0^x e^{-\kappa(x-s)}\theta(s+z_2)\,ds \\
\partial_{z_1} G(z,x) &= e^{-\kappa x} \\
\partial_{z_2} G(z,x) &= \kappa \int_0^x e^{-\kappa(x-s)} \partial_x \theta(s+z_2)\,ds \\
&= \kappa \left| e^{-\kappa(x-s)}\theta(s+z_2) \right|_{s=0}^{x} - \kappa^2 \int_0^x e^{-\kappa(x-s)}\theta(s+z_2)\,ds \\
&= \kappa\theta(x+z_2) - \kappa\theta(z_2)e^{-\kappa x} - \kappa^2 \int_0^x e^{-\kappa(x-s)}\theta(s+z_2)\,ds \ .
\end{aligned}$$

Sorting the $e^{-\kappa x}$-terms yields indeed that

$$\mu_1(z) = -\kappa(z_1 - \theta(z_2)) \ .$$

As in example 3.3, the volatility term can be specified freely. Positivity of the process Z^1 with the volatility structure given above is guaranteed for each $T < \infty$, since Z^1 dominates the solution to

$$dy_t = \kappa(\theta^* - y_t)\,dt + \nu(t)\, y_t^\alpha \, dW_t^1 \qquad \theta^* := \inf{}_{t \le T} \theta(t) > 0$$

(see comparison theorems for stochastic coefficients in Protter [P04] pg. 324).

To show that (3.22) fits the market, note that

$$\begin{aligned}
G(v_0(0), 0; x) &= v_0(0)e^{-\kappa x} + \kappa \int_0^x e^{-\kappa(x-s)}\theta(s)\,ds \\
&= v_0(0)e^{-\kappa x} + \kappa \int_0^x e^{-\kappa(x-s)} \left(v_0(s) + \frac{1}{\kappa}\partial_x v_0(s) \right) ds \\
&= v_0(0)e^{-\kappa x} + \kappa e^{-\kappa x} \int_0^x e^{\kappa s} v_0(s)\,ds \\
&\quad + \left| e^{-\kappa(x-s)} v_0(s) \right|_{s=0}^{x} - \kappa e^{-\kappa x} \int_0^x e^{\kappa s} v_0(s)\,ds \\
&= v_0(x) \ ,
\end{aligned}$$

as claimed. □

A notable advantage of the previous approach is that if applied to Heston's model, the stock price retains its martingale property as long as the correlation ρ remains non-positive. Moreover, its Fourier transform can be computed relatively efficiently as is discussed in [BBFJLO06].

Part II

Hedging

Chapter 4

Theory of Replication

In part I of this thesis, we have introduced variance curve models where the price processes of variance swaps and the stock price were at least local martingales under a measure $\mathbb{P}$, so we do not face questions of arbitrage.

However, we have frequently expressed our desire to use a variance curve model to compute hedges for exotic payoffs with respect to stock and variance swaps. This of course requires that the market under consideration is complete. In this chapter, we will therefore discuss conditions under which both general Markov-driven models and our variance curve models are extremal on their filtrations (an extension to these results can be found in Buehler/Teichmann [BT06]).

In the next chapter, we shall also discuss the practical issue of actually implementing a hedging strategy: even if a model is a good picture of reality, we cannot expect it to fit to the market perfectly. Hence, we will need to *recalibrate* not only the states, but also the allegedly constant parameters of the model.[1] To this end, we will develop the concept of the *meta-model* of an institution and will show the impact on recalibration. In particular, we will show that changing certain parameters such as the speed of mean-reversion in Heston will result in arbitrage in the "meta-model" of the institution.[2]

4.1 Problem Statements and Overview

The first step we will take is to clarify that we do not intend to hedge all possible payoffs which are measurable with respect to the "big" filtration $\mathbb{F}$. To this end, let us take a step back and consider a general market with traded instruments $S = (S^1, \dots, S^N)$ defined on the stochastic base $\mathbb{W} = (\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ which supports an $\mathbb{F}$-extremal Brownian motion $W = (W^1, \dots, W^d)$.

In such a market, we will usually only want to attempt to perfectly hedge contracts which depend on the tradable instruments only. Mathematically, this means that a potential payoff H_T is measurable with respect to the filtration generated by S; we accordingly call this market the *market of relevant payoffs*, denoted by

$$L^1_+(\mathcal{F}^S_T; \mathbb{P}) := \left\{ H_T \geq 0 \,\middle|\, H_T \in L^1(\mathcal{F}^S_T; \mathbb{P}) \right\} \tag{4.1}$$

(we will drop the notion of $\mathbb{P}$ if it is obvious from the context).

[1] The difference between a *state* and a *parameter* of a model is that a state (such as "ShortVol" ζ_t in Heston's model in example 3.3) has some prescribed random dynamics, while a parameter (for example, Heston's speed of mean-reversion κ) is supposed not to change over the life of the trade.

[2] Please refer to chapter 5.1 for precise definitions.

Remark 4.1 *Payoffs which are "not relevant" in the above sense are contracts on non-tradable underlying quantities such as the weather, electricity etc. The adjective "not relevant" refers to the fact that we can not expect to perfectly hedge such payoffs. They may well be very relevant in other markets.*

The question of completeness in such a market is a question of replication:

Definition 4.2 (Market completeness) *Let $S = (S_t)_{t\geq 0}$ with $S_t = (S_t^1, \dots, S_t^N)$ be a vector of local martingales.*

- *Let $\mathcal{A}$ be a σ-algebra such that $\mathcal{A} \subseteq \mathcal{F}_T$ for some $T < \infty$. We say* the market $L^1_+(\mathcal{A})$ is complete with respect to S *if each $H_T \in L^1_+(\mathcal{A})$ can be written as*

$$H_T = H_0 + \sum_{k=1}^{N} \int_0^T \Delta_u^k \, dS_u^k$$

for some $\Delta \in L_T^{loc}(S; \mathbb{F}^S)$ and $H_0 = \mathbb{E}[\, H_T\,] \in \mathbb{R}_{\geq 0}$ such that the value process $H = (H_t)_{t\in[0,T]}$ defined as

$$H_t := H_0 + \sum_{k=1}^{N} \int_0^t \Delta_u^k \, dS_u^k \ . \tag{4.2}$$

remains non-negative. We then say that Δ "replicates" or "hedges" the payoff H_T. The constant H_0 is called the price *of H_T.*

- *Let $\mathbb{A} = (\mathcal{A}_t)_{t\geq 0}$ be a sub-filtration of $\mathbb{F}$. We then say that* the market $\mathbb{A}$ is complete with respect to S *if $L^1_+(\mathcal{A}_T)$ is complete with respect to S for all finite T.*

Obviously, market completeness with respect to S is essentially the *predictable representation property* (PRP) of the vector S, except for the non-negativity requirement of the value process. This limitation is required to avoid "suicide strategies". The non-negativity requirement also ensures that all value processes (4.2) are true martingales.[3]

We also remark that if a hedging strategy exists, it is unique in $L^{\text{loc}}(S)$.

Problem (**P4**)
When is the market $\mathbb{F}^S$ of relevant payoffs complete with respect to S ?

What does "completeness" mean in real markets? In practise, the replication of a payoff H_T is thought to be the result of "delta-hedging": here, we use the derivatives of the value function of the payoff with respect to the coordinates of the tradable instruments as hedging ratios.

The basic idea is as follows: assume as above that we observe a vector $S = (S^1, \dots, S^N)$ of tradable assets such that S is a local martingale, and additionally assume that S is *Markov*. Let then H_T be some $\mathcal{F}_T^S$-measurable payoff such that $H_T \equiv H(S_T)$ for some measurable, non-negative function H. We can then obviously define the non-negative martingale $H = (H_t)_{t\in[0,T]}$ via

$$H_t := \mathbb{E}\left[\, H_T \mid \mathcal{F}_t^S \,\right] \ .$$

[3]It is clear that H_t is at least a local martingale, so it is a super-martingale because it is bounded from below (cf. footnote page 24). Hence, $\mathbb{E}[\, H_T\,] \leq \mathbb{E}[\, H_0\,] = \mathbb{E}[\, H_T\,]$, which proves that H is actually a martingale. See also section 4.2.2 where we discuss issues arising from pricing and hedging in a situation when the stock price which is only a local martingale.

Due to the Markov-property of S, there exists then a function h_t such that

$$H_t \equiv h(t;S_t) := \mathbb{E}[\, H_T \mid S_t \,] \ .$$

If $h(t;s)$ is now differentiable in t and twice differentiable in s, then we can apply Ito and obtain

$$\begin{aligned} H_T &= H_0 + \sum_{k=1}^N \int_0^T \partial_{s^k} h(t;S_t)\, dS_t^k + \left\{ \int_0^T \partial_t h(t;S_t)\, dt + \frac{1}{2} \sum_{j,k=1}^n \int_0^T \partial^2_{s^k s^j} h(t;S_t)\, d\langle S^k, S^j \rangle_t \right\} \\ &= H_0 + \sum_{k=1}^N \int_0^T \partial_{s^k} h(t;S_t)\, dS_t^k \ , \end{aligned}$$

where the drift terms vanish since $(H_t)_t$ is by construction a martingale.

This shows that "delta-hedging works" in the sense that we can replicate H_T by the "classic" hedge in terms of S via the derivative of its value function h with respect to the tradable instruments.

General Markovian Markets

Clearly, the above approach requires that h is sufficiently differentiable. Moreover, it does not deal with arbitrary measurable functions $H_T \in L^1_+(\mathcal{F}^S_T)$. Finally, it also requires that all information in the market is carried by the vector S. However, we will want to write contracts on the realized variance of the stock price, which in itself is not a local martingale, but observable in the market.

To allow the incorporation of additional information, we therefore assume that not necessarily $S = (S^1, \dots, S^N)$ itself, but a vector $Y = (S, A)$ is jointly Markov, where $A = (A^1, \dots, A^M)$ is a left-continuous process of finite variation. Note that A is not tradable.

We approach the issue of smoothness in section 4.2 by assuming that the mapping $P^{S,A} : C^0(\mathbb{R}^{N+M}) \to C^0(\mathbb{R}^{1+N+M}_{\geq 0})$ given as

$$P^S(\mathbb{H})(t,s) := \mathbb{E}[\, \mathbb{H}(S_t) \mid S_0 = s \,]$$

weakly preserves smoothness (cf. definition 4.3) in the sense that

$$P^{S,A}(C^\infty_K) \subseteq C^{0,1,0}\Big(\mathbb{R}_{\geq 0} \times \mathbb{R}^N_{\geq 0} \times \mathbb{R}^M_{\geq 0} \Big)$$

where C^∞_K is the space of all smooth functions whose derivatives all have compact support. The assumption means that the value function for very smooth payoffs is continuous and differentiable in the S-coordinates. The latter derivatives are going to be our "deltas".

Indeed, it is shown in theorem 4.4 that under the above assumption, the market is complete, i.e. that all non-negative payoffs H_T which are measurable with respect to $\mathcal{F}^S_T$ can be replicated using S, i.e. there exists a $\Delta \in L^{\mathrm{loc}}_T(S)$ and a $H_0 \in \mathcal{F}^S_0$ such that

$$H_T = H_0 + \sum_{k=1}^N \int_0^T \Delta^k_u \, dS^k_u \ .$$

Moreover, if the value process $H_t = \mathbb{E}[\, H_T \mid \mathcal{F}_t \,]$ is a C^1-function $h(t;S_t) = H_t$ of S, then "delta hedging works" in the sense that

$$\Delta^k_t = \partial_{s^k} h(t;S_t) \ .$$

These results are also discussed in Buehler/Teichmann [BT06], where we also comment on the impact of this observation for the completeness of "infinite dimensional" models in Hilbert spaces as those discussed in section 2.4, the first obvious result being that as long as such a model admits an FDR in terms of tradable market instruments, then it is complete.

Complete Variance Swap Markets

The previous results can not directly be applied to our variance curve framework, because we face infinitely many tradable variance swaps. It is therefore necessary to ensure that a finite selection of variance swaps (and the stock) is sufficient to hedge an exotic product. This is done in section 4.2.3.

PROBLEM (**P5**)
Given the strong Markov variance curve model (G, Z, ρ), under which circumstances is the market complete?

Theorem 4.19 shows that the market is complete if the variance curve is sufficiently invertible and if a suitable finite set of variance swaps plus the stock price weakly preserve smoothness. In short, "sufficiently invertible" means that the variance swap price

$$\mathbb{G}(z, x) := \int_0^x G(z, y)\, dy$$

can be inverted in z for a finite set of maturities which can "shift" in time. We call this property τ-*Invertibility*. It is defined in definition 4.17 below.

4.2 Hedging in Complete Markets

In this section, we will prove that under relatively weak assumptions, a strong Markov variance curve market model creates a complete market in which all exotic payoffs can be replicated by positions in stock and a finite number of variance swaps.

Our approach is as follows: first, we shall prove with theorem 4.4 a rather general result on complete markets where the traded assets plus some adapted processes of finite variation (such as running variance) are Markov. In this case, the crucial condition to achieve market completeness is "weak preservation of smoothness" (definition 4.3).

Secondly, we will assume that the parameter process Z of a variance curve model can be recovered from observed variance swap prices by an inversion of (the integral of) the variance curve function G and apply the results of the first step to show that any payoff in the sense defined above can be replicated through dynamic trading in stock and variance swaps.

We start with the promised general result on complete markets in a Markovian setting. These results are deepened and discussed further in Buehler/Teichmann [BT06].

4.2.1 General Complete Markovian Markets

Let $\mathbb{W} = (\Omega, \mathcal{F}_\infty, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with a stochastic base with right-continuous, completed filtration which supports an extremal d-dimensional Brownian motion $W = (W^1, \dots, W^d)$.

In this subsection, assume that the market consists of N tradable instruments $S = (S^1, \dots, S^N)$ which are non-negative local martingales. We further stipulate that there is an M-dimensional left-continuous $\mathbb{F}^S$-adapted process $A = (A^1, \ldots, A^M)$ which has finite variation such that the joint process $Y = (S, A)$ is a non-explosive diffusion which uniquely solves an SDE

$$dY_t = \mu(Y_t)\,dt + \sum_{j=1}^{d} \Sigma^j(Y_t)\,dW_t^j \tag{4.3}$$

for locally Lipschitz vectors $\mu, \Sigma^1, \ldots, \Sigma^d$.[4] Evidently, $Y = (S, A)$ is Markov. Also note that $\mathbb{F}^S = \mathbb{F}^{S,A}$, but in the sequel, we will still write $\mathbb{F}^{S,A}$ to stress that payoffs may well depend on the values of A: this process can be used to encode information of the past into the vector (S, A). For example, the realized variance of one of the traded assets is a left-continuous process of finite variation.

Let $X = (X^1, \ldots, X^m)$ be an arbitrary non-explosive diffusion which uniquely solves an SDE

$$dX_t = m(X_t)\,dt + \sum_{k=1}^{d} \nu^k(X_t)\,dW_t^k \ .$$

For non-negative, measurable functions $\mathbb{H} : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ define the operator

$$P^X(\mathbb{H})(t, x) := \mathbb{E}\left[\, \mathbb{H}(X_t) \mid X_0 = x \,\right] \ . \tag{4.4}$$

Definition 4.3 (Weak preservation of smoothness) *We then say that* X weakly preserves smoothness *if there exists some* $T^* > 0$ *such that*

$$P^{S,A}(C_K^\infty(\mathbb{R}^m)) \subseteq C^{0,\,\iota_1,\ldots,\iota_m}\Big((0, T^*] \times \mathbb{R} \times \cdots \times \mathbb{R}\Big)$$

where we set $\iota_i = 0$ *if* $\nu_i^1 = \cdots = \nu_i^1 = 0$ *or* $\iota_i = 1$*, otherwise for* $i = 1, \ldots, m$.

In other words, whenever $\langle X^i \rangle_{T^*} \not\equiv 0$ *for some* $i = 1, \ldots, m$*, then* $x^i \mapsto P^X(\mathbb{H})(t, x^1, \ldots, x^m)$ *for* $t \in (0, T^*]$ *must be at least once differentiable in this coordinate. Otherwise, it is sufficient if* $P_t^X(\mathbb{H})$ *is continuous in this coordinate.*

Translated to our setup, we see that the vector $Y = (S, A)$ *weakly preserves smoothness* iff

$$P^{S,A}(C_K^\infty(\mathbb{R}^{N+M})) \subseteq C^{0,1,0}\Big((0, T^*] \times \mathbb{R}_{\geq 0}^N \times \mathbb{R}^M\Big) \ ,$$

i.e. if for all $H \in C_K^\infty$, then $P^{S,A}(\mathbb{H})(t, s, a)$ is at least once differentiable in s and continuous in t and a.

This property is the key for completeness.

Theorem 4.4 (Completeness) *If* (S, A) *weakly preserves smoothness, then "delta hedging works", i.e. the market* $L_+^1(\mathbb{F}^{S,A})$ *is complete with respect to* S.

Note that completeness is not limited up to the time T^*, up to which weak preservation of smoothness holds. The reason is quite simple: assume that we have shown that the market $L_+^1(\mathcal{F}_{T^*}^{S,A})$ is complete with respect to (S, A). Then, using the Markov property and time-homogeneity of (S, A), we have

$$\mathbb{E}\left[\, \mathbb{H}(S_{2T^*}, A_{2T^*}) \mid (S, A)_{T^*} = (s, a) \,\right] = P^{S,A}(\mathbb{H})(T^*; s, a) \ ,$$

[4] Extensions for the case where (4.3) holds only up to a strictly positive stopping time τ are straight forward.

hence iterated application of the representation up to T^* yields a representation for all payoffs $\mathbb{H}(S_{2T^*}, A_{2T^*}) \in L^1_+(\mathcal{F}^{S,V}_{2T^*})$. See the proof of lemma 4.9 for details.

Before we proceed, we want to state the following essential proposition which gives a sufficient condition for weak preservation of smoothness:

PROPOSITION 4.5 *Assume that μ and $\Sigma^1, \dots, \Sigma^d$ are locally Lipschitz and once differentiable with locally Lipschitz derivatives.*[5] *Then, the process Y weakly preserves smoothness.*

The essence of the previous theorem and its proposition 4.5 is that if the coefficients of (4.3) are sufficiently smooth, then the market $L^1_+(\mathcal{F}^Y)$ remains complete even if the volatility matrix Σ has singularities: in fact, this matrix can be very sparse under the conditions of theorem 4.4 and it can, in contrast to standard results, become zero depending on the value of the finite variation process A (i.e., time or an aggregated quantity such as realized variance). An non-exotic example of such a situation would be an option whose payoff depends on a number of stocks in the same currency which have different trading days (for example, consider a pan-European EUR denoted basket). In such a case, the volatility matrix is singular whenever one of the stocks is not traded.[6] It is also a natural formulation in the sense that the volatility matrix Σ and the drift μ are often smooth in the interior of the domain of the process Y.

In contrast, classical results require a parabolic PDE associated to the SDE (4.3), which in particular means that we require as many tradable assets as we have underlying Brownian motions. For example, Janson/Tysk [JT06] (theorem 2.7) show that

THEOREM 4.6 *Assume that $S = (S^1, \dots, S^d)$ solves the SDE*

$$dY_t = \sum_{j=1}^{d} \Sigma^j(t; S_t)\, dW^j_t$$

with a continuous and locally Lipschitz matrix $\Sigma(t;\cdot)$ which has full rank for all $t \le T^*$. Also assume that $\|\Sigma(t;x)\| \le D(1 + \|x\|)$ on $[0,T^*] \times \mathbb{R}^d$ for some constant D.

Then, $P^S(H)(t;y) := \mathbb{E}[\, \mathbb{H}(S_t) \mid S_0 = y\,]$ is $C^{0,2}$ for all $\mathbb{H}$ such that $\mathbb{H}(S_T) \in L^1_+(\mathcal{F}^S_{T^*})$.

In other words, S weakly preserves smoothness.

Further results in this direction can be found in Janson/Tysk [JT04] (e.g. theorem A.13).

Note that Janson/Tysk [JT06] also specialize their result to cases where the coordinates S^i can be absorbed at the boundary, as it is the case for a CEV process.[7] However, this setting does not cover the case where the matrix Σ is only “temporarily” singular, neither does their setting allow that the volatility of a coordinate S^i becomes zero if $S^i > 0$. In particular, it means that over $\mathbb{R}^m_{>0}$, there must always be as many tradeable assets as driving Brownian motions.

Here is an example of an (admittedly degenerated) “diffusion” which is not weakly preserving smoothness:

EXAMPLE 4.7 *The solution to the deterministic equation*

$$dY_t = Y_t 1_{Y_t \ge 1}\; dt$$

[5] Recall that Y is assumed not to explode.

[6] In this notation, the matrix would not be continuous in t, but this can be approximated by a tight linear function.

[7] The CEV process follows the diffusion $dX_t = X^\alpha_t\, dW^1_t$ and has a unique, non-explosive solution for $1/2 \le \alpha \le 1$. It is absorbed in zero for $\alpha < 1$.

is $Y_t = Y_0\,(1_{Y_0<1} + 1_{Y_0\geq 1}\,t)$. *Consequently,* P^Y *is not preserving smoothness weakly.*

We begin with the proof of proposition 4.5, and will then work towards the proof of theorem 4.4 via a few lemmata.

Proof of proposition 4.5– To highlight the dependency of Y on its origin, y, we now write Y^y. Under the assumptions of the proposition, the map

$$y \longmapsto Y_t^y(\omega)$$

is in C^1 for almost all (t,ω), and there exists a continuous "derivative" process $Z^{(y,i)} = (Z^{(y,i),1},\dots,Z^{(y,i),m})$ for all $i = 1,\dots,m$ such that $Z_t^{(y,i)} = \partial_{y^i} Y_t^y$ almost surely. The vector $Z_t^{(y,i)} = (Z_t^{(y,i)1},\dots,Z_t^{(y,i)m})$ satisfies

$$dZ_t^{(y,i)} = \sum_{k=1}^m Z_t^{(y,i)k} \left\{ \partial_{y^k}\mu(Y_t)dt + \sum_{j=1}^d \partial_{y^k}\sigma^j(Y_t)dW_t^j \right\} \tag{4.5}$$

(cf. Protter [P04] theorem 39, pg. 305). It is defined up to the same explosion time as Y^y, which in our current setting is infinite.[8]

Let now $\mathbb{H} \geq 0$ be some C_K^∞ function and fix some $t > 0$. Since $\partial_{y^i}\mathbb{H}$ has compact support, say $D \subseteq \mathbb{R}^m$, its support is bounded, hence the continuous function $\partial_{y^i}\mathbb{H}(Y_t^y)$ is bounded by the constant $K = \|\mathbb{H}\|_\infty$, independently of y. Moreover, it vanishes outside D. Since $Z^{y,i}$ is continuous, it is bounded on D, such that the product $Z_t^{(y,i)}\partial_{y_i}\mathbb{H}(Y_t^y)$ is bounded. Thus, the limit of the derivative can be taken out of the expectation and we find that

$$\partial_{y^i} P^Y(H)(t;y) := \mathbb{E}\left[\,\mathbb{H}(Y_t^y)\,\right]$$

is indeed C^1 in y. Continuity in t follows from the Feller property of the diffusion Y. □

We now proceed with the proof of theorem 4.4 in the current diffusion setting. For more general cases, see Buehler/Teichmann [BT06], where the conditions of proposition 4.5 are further relaxed, too.

LEMMA 4.8 *Fix* $T \leq T^*$ *and assume* $S \in \mathcal{H}^2$. *If* $H_T = \mathbb{H}(S_T, A_T) \geq 0$ *for a* $\mathbb{H} \in C_K^\infty$ *such that*

$$h(t;s,a) := \mathbb{E}\left[\,\mathbb{H}(S_T,A_T)\mid S_t = s, A_t = a\,\right]$$

is C^1 *in its* s*-argument and continuous in* (t,a)*, then* $\Delta = (\Delta^1,\dots,\Delta^M)$ *with*

$$\Delta_t^k := \partial_{S^k} h(t;S_t,A_t) \tag{4.6}$$

is an $\mathbb{F}^{S,V}$*-predictable element of* $L_T^2(S)$*, and we have*

$$\mathbb{H}(S_T,A_T) = H_0 + \sum_{k=1}^M \int_0^T \Delta_t^k\, dS_t^k$$

in $L^2(\mathbb{P})$ *where* $H_0 = \mathbb{E}\left[\,\mathbb{H}(S_T,A_T)\,\right]$. *The value process* $(H_t)_{t\in[0,T]}$ *is a non-negative martingale.*

[8] To see that (4.5) has a non-exploding solution, fix y and let $\tau_n := \inf\{t : ||Y_t|| \geq n\}$. Then, the stopped processes Y^n are bounded and so are the coefficients on the right hand side of (4.5), which in turn means that up to each τ_n, $z \mapsto z\partial_{y^i}\mu(Y_t)$ and $z \mapsto z\partial_{y^i}\Sigma^j(Y_t)$ are globally Lipschitz, which implies that a solution for (4.5) exists up to each τ_n. Taking the limit yields that $Z^{(i)}$ is everywhere well-defined. See the proof of theorem 39 pg. 305 in Protter [P04] for more details.

Proof – Since $\mathbb{H}$ has compact support and is continuous, it is bounded. Hence, $H_t = h(t; S_t, A_t)$ is a non-negative bounded martingale. For $m := N + M + 1$, we choose a "Dirac sequence" $\varphi_n : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$, $n = 1, \ldots$ of non-negative smooth $L^\infty(\lambda^m)$-functions with compact support and $\int_{\mathbb{R}^m} \varphi_n(x)\, dx = 1$ for all n. Define

$$h_n(t; s, a) := (\varphi_n \star h)(t; s, a) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^M} \int_0^T h(t'; s', a') \varphi_n(t - t', s - s', a - a')\, dt'\, da'\, ds' \ . \quad (4.7)$$

For each $n \in \mathbb{N}$, the function h_n is smooth in all m parameters and we have $h_n \to h$ and $\partial_{S^k} h_n \to \partial_{S^k} h$ as $n \uparrow \infty$ in $L^p(\lambda^m)$ for all $p \in [1, \infty)$. Since h is bounded, convergence also holds for $p = \infty$. Hence, $h_n \to h$ in $L^\infty(\lambda^m)$ such that $h_n(t; S_t, A_t) \to h(t; S_t, A_t)$ as $n \uparrow \infty$ in $L^2(\mathbb{P})$:

$$\mathbb{E}\left[(h_n(t; S_t, A_t) - h(t; S_t, A_t))^2 \right] \leq \|h_n - h\|_\infty^2 \downarrow 0 \ .$$

Next, note that

$$\begin{aligned} h_n(t; S_t, A_t) - h(0; S_0, A_0) \quad &= \quad \sum_{k=1}^M \int_0^t \partial_{S^k} h_n(u; S_u, A_u)\, dS_u^k + \sum_{i=1}^N \int_0^t \partial_{A^i} h_n(u; S_u, A_u)\, dA_u^i \\ &\quad + \int_0^t \partial_t h_n(u; S_u, A_u)\, du + \frac{1}{2} \sum_{k,\ell=1}^M \int_0^t \partial_{S^k S^\ell} h_n(u; S_u, A_u)\, d\langle S^k, S^\ell \rangle_u \ . \end{aligned}$$

We have shown before that the left hand side converges in $L^2(\mathbb{P})$ against the L^2-martingale $h(t; S_t, A_t) - h(0; S_0, A_0)$, hence the drift terms on the right hand side of the above equation must vanish such that

$$h(t; S_t, A_t) - h(0; S_0, A_0) = \lim_{n \uparrow \infty} \sum_{k=1}^M \int_0^t \partial_{S^k} h_n(u; S_u, A_u)\, dS_u^k \ ,$$

where the limit is taken in $L^2(\mathbb{P})$. It remains to show (4.6), i.e. that the hedging ratios converge against the hedging strategy Δ and that it is an element of $L_T^2(S)$. To this end, let $\tau_\ell := \inf\{t : \|S_t\|_2^2 + \|A_t\|_2^2 > \ell\}$ for $\ell \in \mathbb{N}$. On $\{t \leq \tau_\ell\}$, the process (S, A) is bounded and we have

$$\begin{aligned} &\mathbb{E}\left[\int_0^{\tau_\ell \wedge T} \Big(\partial_{S^k} h_n(u; S_u, A_u) - \partial_{S^k} h(u; S_u, A_u) \Big)^2 d\langle S^k \rangle_u \right] \\ \leq \ &\mathbb{E}\left[\int_0^{\tau_\ell \wedge T} \Big(\sup_{t,s,a : \|s\|_2^2 + \|a\|_2^2 \leq \ell} |\partial_{S^k} h_n(t; s, a) - \partial_{S^k} h(t; s, a)| \Big)^2 d\langle S^k \rangle_u \right] \quad \longrightarrow 0 \quad (n \uparrow \infty) \ . \end{aligned}$$

Hence, on $t \leq \tau_\ell$ (4.6) is satisfied. Therefore,

$$\lim_{n \uparrow \infty} \partial_S h_n(u; S_u, A_u) = \partial_S h(t; S_t, A_t) =: \Delta_t$$

such that $\Delta \in L_T^{\text{loc}}(S)$. Moreover, using our previous result that $h(t; S_t, A_t) \in L^2(\mathbb{P})$, we have

$$\mathbb{E}\left[\sum_{k=1}^M \int_0^t (\Delta_t^k)^2\, d\langle S^k \rangle_u \right] = \mathbb{E}\left[\Big(h(t; S_t, A_t) - h(0; S_0, A_0) \Big) \right] < \infty \ ,$$

i.e. $\Delta \in L_T^2(S)$. Non-negativity of the value process is guaranteed by construction. $\square$

LEMMA 4.9 *Fix $T < \infty$ (possibly larger than T^*) and assume $S \in \mathcal{H}^2$. Let $\mathbb{H} : \mathbb{R}^{M+N} \to \mathbb{R}_{\geq 0}$ be a non-negative measurable function such that $H_T := \mathbb{H}(S_T, A_T)$ is in $L^2(\mathbb{P})$. Then, there exists a $\Delta \in L^2_T(S)$ such that*

$$H_T = H_0 + \sum_{k=1}^{M} \int_0^T \Delta_t^k \, dS_t^k \tag{4.8}$$

where $H_0 = \mathbb{E}\,[\,H_T\,]$.

Moreover, the value process $H_t := H_0 + \sum_{k=1}^M \int_0^t \Delta_u^k \, dS_u^k$ is a non-negative martingale.

Proof – First, assume that $T \leq T^*$.

Let $m := M + N$ and choose again a Dirac sequence $\varphi_n : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ for $n = 1, \dots$ of non-negative smooth $L^\infty(\lambda^m)$ functions with common compact support and $\int_{\mathbb{R}^m} \varphi_n(x)\,dx = 1$ for all n. Define the C_K^∞ functions

$$\mathbb{H}^n(s, a) := (\varphi_n \star \mathbb{H})(s, a)$$

as in (4.7). According to the previous lemma 4.8, $h^n(t; s, a) := \mathbb{E}\,[\,\mathbb{H}^n(S_T, A_T) \mid S_t = s, A_t = a\,]$ is then C^1 for all n, and we have in $L^2(\mathbb{P})$ the martingale representation

$$\mathbb{H}^n(S_T, A_T) = h^n(0; S_0, A_0) + \sum_{k=1}^{M} \int_0^T \partial_{S^k} h^n(t; S_t, A_t) \, dS_t^k \ .$$

Since $\mathbb{H}^n \to \mathbb{H}$ in $L^\infty(\lambda^n)$ it follows as before that the right hand side converges in $L^2(\mathbb{P})$. Therefore, the vector $(\partial_{S^1} h^n(t; S_t, A_t), \dots, \partial_{S^M} h^n(t; S_t, A_t))$ converges as above in $L^2_T(S)$ to some $\mathbb{F}^{S,A}$-predictable $\Delta \in L^2_T(S)$ for which (4.8) above holds. Non-negativity of the value process follows from the non-negativity of the value processes for $\mathbb{H}^n(S_T, A_T)$.

As a next step, assume that $T \in (T^*, 2T^*]$. Let $t := T - T^*$. Because of the Markov property of (S, A), we have

$$\mathbb{E}\,[\,\mathbb{H}(S_{T^*+t}, A_{T^*+t}) \mid S_{T^*} = s\; A_{T^*} = a\,] = P^{S,A}(\mathbb{H})(t; s, a)$$

for all $t \in [0, T^*]$. Let us denote by S^s and A^a the processes S and A started in s and a, respectively. Our previous results show that there is some $\Delta \in L^2_t(S^a)$ such that

$$\mathbb{H}(S_t^s, A_t^a) = P^{S,A}(\mathbb{H})(t; s, a) + \sum_{k=1}^{M} \int_0^t \Delta_u^k \, dS^{s\,k}_{\ u} \ .$$

Now observe that $\mathbb{H}^2 := P^{S,A}(\mathbb{H})(t; S_{T^*}, A_{T^*}) \in L^2(\mathcal{F}_{T^*}^{S,A})$, i.e. we apply again our previous results to find that there exists some $\Delta' \in L^2_{T^*}(S)$ such that

$$\mathbb{H}^2 = \mathbb{E}\,\left[\,\mathbb{H}^2\,\right] + \sum_{k=1}^{M} \int_0^t \Delta'^k_u \, dS_u^k \ .$$

Joining the two results yields that

$$\mathbb{H}(S_{T^*+t}, A_{T^*+t}) = \mathbb{E}\,[\,\mathbb{H}(S_{T^*+t}, A_{T^*+t})\,] + \sum_{k=1}^{M} \int_0^t \tilde{\Delta}_u^k \, dS_u^k$$

for the appropriate $\tilde{\Delta} \in L^2_T(S)$. □

LEMMA 4.10 *Fix $T < \infty$ and assume $S \in \mathcal{H}^2$. Let $\mathbb{H} : (\mathbb{R}^{M+N})^n \to \mathbb{R}_{\geq 0}$ be a non-negative measurable function such that $H_T := \mathbb{H}(S_{T_1}, A_{T_1}; \ldots ; S_{T_n}, A_{T_n})$ for $0 = T_0 < T_1 < \cdots < T_n := T < \infty$. is in L^2. Then, there exists a $\Delta \in L^2_T(S, \mathbb{F}^{S,A})$ such that (4.8) holds.*

Proof – Assume w.l.g. that $T_{\ell-1} < t \leq T_\ell$. Then, the same procedure as in the proofs for the previous lemmata can be applied on $(T_{\ell-1}, T_\ell]$ to the measurable function

$$(s, a) \longmapsto \mathbb{E}\left[\, \mathbb{H}(S_{T_1}(\omega), A_{T_1}(\omega); \ldots ; S_{T_{\ell-1}}(\omega), A_{T_{\ell-1}}(\omega); S_{T_\ell}, A_{T_\ell}; \ldots ; S_{T_n}, A_{T_n}) \,\middle|\, S_t = s, A_t = a \,\right] ,$$

conditional on $\mathcal{F}_{T_{\ell-1}}$. □

LEMMA 4.11 *Fix $T < \infty$, assume $S \in \mathcal{H}^2$ and let $H_T \in L^2(\mathcal{F}^{S,A}_T)$ be non-negative. Then there exists a $\Delta \in L^2_T(S, \mathbb{F}^{S,A})$ such that (4.8) holds.*

Proof – Chose a countable representation $\{t_i\}_{i\in\mathbb{N}}$ of $\mathbb{Q}$. For all n, let $T^n_i := t_i$ for $i = 1, \ldots, n$. Define a complete, discrete-time filtration $\mathbb{G} = (\mathcal{G}_n)_{n\in\mathbb{N}}$ by

$$\mathcal{G}_n := \sigma\left((S, A)_{T^n_1}, \ldots , (S, A)_{T^n_n}\right) \bigvee \mathcal{F}_0 \ .$$

As a next step, consider the discrete-time $\mathbb{G}$-martingale

$$H^n := \mathbb{E}\left[\, H_T \mid \mathcal{G}_n \,\right] = \mathbb{E}\left[\, H_T \mid (S, A)_{T^n_1}, \ldots , (S, A)_{T^n_n} \,\right] \ .$$

Each of the random variables H^n is square integrable and a function of finitely many values of (S, A), hence lemma 4.10 yields a $\Delta^{(n)} = (\Delta^{(n),1}, \ldots , \Delta^{(n),M}) \in L^2(S)$ such that

$$H^n = \mathbb{E}\left[\, H^n \,\right] + \sum_{k=1}^{M} \int_0^T \Delta^{(n),k}_t \, dS^k_t \ .$$

in $L^2(\mathbb{P})$. The martingale $(H^n)_n$ converges in L^2 to H_T, hence Δ^n must also converge in $L^2(\langle S\rangle \otimes \mathbb{P})$ to some $\Delta \in L^2(S)$ such that

$$H_T = \mathbb{E}\left[\, H_0 \,\right] + \sum_{k=1}^{M} \int_0^T \Delta^k_t \, dS^k_t \ ,$$

as desired. □

The following lemma essentially proves theorem 4.4.

LEMMA 4.12 (Replication property of S) *Let $T < \infty$. Assume $S \in \mathcal{H}^{loc}$ and let $H_T \in L^1_+(\mathcal{F}^{S,A}_T)$. Then there exists a $\Delta \in L^{loc}_T(S, \mathbb{F}^{S,A})$ such that*

$$H_T = H_0 + \sum_{k=1}^{M} \int_0^T \Delta^k_t \, dS^k_t \ . \tag{4.9}$$

for $H_0 = \mathbb{E}[\, H_T \,]$, such that the value process $H_t := H_0 + \sum_{k=1}^{M} \int_0^t \Delta^k_u \, dS^k_u$ is a non-negative martingale.

In other words, the market $\mathbb{F}^{S,A}$ is complete with respect to S.

Proof – Let

$$H_t := \mathbb{E}\left[\, H_T \mid \mathcal{F}_t^{S,A} \,\right] \ .$$

Now, choose a localizing sequence $(\tau_n)_n$ such that the stopped processes $S^{(n)}$ and $H^{(n)}$ are in $L^2(\mathbb{P})$, for example by setting $\tau_n := \inf\{t : \|H_t\|_2^2 + \|S_t\|_2^2 \geq n\}$. Note that

$$H_t^{(n)} = H_{\tau_n \wedge t} = \mathbb{E}\left[\, H_T \mid \mathcal{F}_{\tau_n \wedge t}^{S,A} \,\right] = \mathbb{E}\left[\, H_T \mid \mathcal{F}_t^{S^{(n)},A^{(n)}} \,\right]$$

where $A_t^{(n)} := A_{\tau_n \wedge t}$. Hence, for each n, there exists $\Delta^{(n)} = (\Delta^{(n),1}, \ldots, \Delta^{(n),M}) \in L_T^2(S^n)$ such that

$$H_T^n = H_0 + \sum_{k=1}^M \int_0^T \Delta_t^{(n),k}\, dS_t^{(n),k} = H_0 + \sum_{k=1}^M \int_0^{\tau_n \wedge T} \Delta_t^k\, dS_t^k \tag{4.10}$$

where we have (consistently) defined $\Delta_t := \Delta_t^{(n)}$ whenever $\{\tau > t\}$, i.e. $\Delta \in L_T^{\mathrm{loc}}(S)$. In other words, (4.10) is just the local representation (4.9) of H_T we were looking for. We also have proved that $H_0 = \mathbb{E}\left[\, H_T \,\right]$ in (4.9) and that $(H_t)_t$ is a non-negative martingale. □

4.2.2 Pricing with Local Martingales

In the case of a strict local martingale S, the previous result is somehow surprising: it tells us that the payoff $H_T = S_T$ can be realized by charging an initial price of $S_0^* := \mathbb{E}\left[\, S_T \,\right]$ which is strictly lower than today's value S_0 of the stock S (recall that S is a supermartingale): according to lemma 4.12, there is some unique $\Delta \in L_T^{\mathrm{loc}}(S)$ such that

$$S_T = S_0^* + \int_0^T \Delta_t\, dS_t \tag{4.11}$$

holds locally. But what about the obvious representation

$$S_T = S_0 + \int_0^T 1\, dS_t \tag{4.12}$$

which seems to hold, too? It looks as if we can lock-in a riskless profit: the strategy is to short the asset for the gain of S_0 and to invest $S_0^* < S_0$ according to the hedging strategy (4.11). At maturity, we have replicated the position S_T via (4.11), so we are able to serve the obligation from shorting the asset. The risk-free gain appears to be $S_0 - S_0^* > 0$. The value process of this strategy is

$$H_t = (S_0^* - S_0) + \int_0^t (\Delta_u - 1)\; dS_u = \mathbb{E}\left[\, S_T \mid \mathcal{F}_t \,\right] - S_t \ .$$

The crucial point here is that this violates the condition that the value process must be non-negative (or at least bounded from below) in definition 4.2. Indeed, if S is a strictly local martingale, then[9]

$$\lim_{n\uparrow\infty} n\, \mathbb{P}\left[\sup_{t\in[0,T]} S_t \geq n \right] > 0 \ . \tag{4.13}$$

[9]Let $\tau_n := \inf\{t : S_t \geq n\}$ for all $n > S_0$. Then, $S_0 = \mathbb{E}\left[\, S_T^n \,\right] = n\mathbb{P}[\tau_n < T] + \mathbb{E}\left[\, S_T 1_{\tau_n > T} \,\right]$, i.e. by monotone convergence $0 < S_0 - S_0^* = S_0 - \mathbb{E}\left[\, S_T \,\right] = \lim_{n\uparrow\infty} n\mathbb{P}[\tau_n < T] = \lim_{n\uparrow\infty} n\mathbb{P}[\sup_{t\in[0,T]} S_t > n]$.

Cox/Hobson [CH05] offer an intuitive description of (4.13): they point out that the above property essentially says that the value of our short position in S will exceed all bounds with a non-zero probability. Such a liability is clearly not feasible for a real-life financial institution. In mathematical finance, this phenomenon is often explained in terms of "doubling strategies", cf. Karataz/Shreve [KS98] example 2.3 on page 8. Such strategies are not admissible in continuous-time arbitrage-free markets.

In general, using a local martingale as a stock price process can produce many counter-intuitive results. For example, there will also be some strike $K > 0$ such that

$$\mathbb{E}\left[\,(S_T - K)^+\,\right] < (S - K)^+ \ ,$$

i.e. the price of a call is below its intrinsic value.

REMARK 4.13 *Cox/Hobson [CH05] show that the price S_0^* computed above is the fair price which replicates S_T under the requirement to maintain a non-negative value process.*

They also discuss the price of a call under the requirement of always having a position which exceeds the intrinsic value of a claim. (While this property is automatically satisfied if S is a true martingale, it does not hold in the case where S is a strictly local martingale.)

4.2.3 Hedging with Variance Swaps

Theorem 4.4 is a neat result for markets of the specified form, but it can not yet be applied directly to our variance curve framework, since the parameter process Z is not a visible variable on the market.

We will therefore need to rephrase the results according to our previous setup. To this end, recall that we initially aimed at using variance swaps to replicate our payoffs. The slight complication is that we only want to use a finite number of such variance swaps.

Assume that (G, Z, ρ) is a global Markov variance curve model (cf. definition 2.22 on page 30). Hence, the m-dimensional parameter process Z is the unique strong, non-explosive solution to

$$dZ_t = \mu(Z_t)\,dt + \sum_{j=1}^{d} \sigma^j(Z_t)\,dW_t^j$$

and it is consistent with the variance curve functional G, such that the SDE

$$\frac{dS_t}{S_t} = \sqrt{\zeta_t}\,\sum_{j=1}^{d} \rho^j(Z_t, S_t)\,dW_t^j$$

with $\zeta_t := G(Z_t; 0)$ has a unique strong solution $S = (S_t)_{t\geq 0}$, which is a local martingale.

As discussed above, we will not attempt to replicate payoffs which are measurable with respect to the filtration $\mathbb{F}$ which is generated by the underlying driving Brownian motion W. Even though this approach is used widely in the financial literature, we rather consider the *market of relevant payoffs* with payoffs whose value at maturity can be deduced by observing the values of traded assets during the life of the contract: in our case, these are the stock price and the variance swap prices.

For this reason, let $\mathbb{F}^V = (\mathcal{F}_t^V)_{t\geq 0}$ be the filtration generated by the variance swaps prices, i.e.

$$\mathcal{F}_t^V = \sigma\Big(V_u(T_1), \ldots, V_u(T_\ell); u \leq t; \ell \in \mathbb{N}; 0 \leq T_1 < \cdots < T_\ell < \infty\Big) \ .$$

This filtration describes the history of the pure variance processes. This definition is meaningful because $V_t(\cdot)$ is continuous.

Let us also define $\mathbb{F}^{S,V} := (\mathcal{F}_t^{S,V})_{t\geq 0}$ as the market filtration jointly generated by the variance swap prices and the stock price, i.e.

$$\mathcal{F}_t^{S,V} := \mathcal{F}_t^V \bigvee \mathcal{F}_t^S \ . \tag{4.14}$$

As in the previous section, the filtration $\mathbb{F}^{S,V}$ describes what we can "see" from observing the available tradable assets. Note, in particular, that both the quadratic variation $\langle \log S \rangle$ and the short variance process $\zeta_t = \partial_t^- \langle \log S \rangle_t$ are adapted to $\mathbb{F}^{S,V}$.

We frequently spoke of "options on variance" and "options on realized variance". Here are the formal definitions:

Definition 4.14 (Option on Variance) *An* option on variance *with maturity T is a payoff $H_T \in L^1_+(\mathcal{F}_T^V)$.*

Definition 4.15 (Option on Realized Variance) *An* option on realized variance *H_T (also called a* vanilla *option on variance) with maturity T is given in terms of a measurable function $\mathbb{H} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that*

$$H_T := \mathbb{H}\left(\int_0^T \zeta_t \, dt\right)$$

is an option on variance.

In the case of variance swaps, our previous definition 4.2 of completeness is not wide enough: in contrast to the setting in the previous section, we now have an infinite number of tradable instruments available for hedging. But in practise, hedging with an infinite number of securities is clearly not an applicable approach. The idea is that we will never want to invest in more than finitely many variance swaps.[10]

Definition 4.16 (Completeness for infinitely many tradable assets) *Let $S = (S^\imath)_{\imath\in\mathcal{I}}$ for $\mathcal{I} \subseteq \mathbb{R}$ be a not necessarily countable family of local martingales.*

Let $\mathcal{A} \subseteq \mathcal{F}_T$ as above. Then, we say the market of payoffs $L^1_+(\mathcal{A})$ is complete with respect to S *if there exist finitely many indices $\imath_1, \ldots, \imath_\ell$ (dependent on T) such that the market $L^1_+(\mathcal{A})$ is complete with respect to $(S^{\imath_1}, \ldots, S^{\imath_\ell})$.*

Completeness of a filtration $\mathbb{A} = (\mathcal{A}_t)_{t\geq 0}$ is defined similarly.

From our previous discussion it is clear that we aim to find for all $T < \infty$ a finite number of maturities $T_1, \ldots, T_m$ such that we can use the respective variance swaps and the stock to replicate our payoffs. To formalize our approach, consider the *variance swap price functional*

$$\mathbb{G}(z; x) := \int_0^x G(z; y) \, dy \tag{4.15}$$

which computes the value of a variance swap with time-to-maturity x given the state parameter z. This function is $C^{2,2}$. The idea is to invert the mapping $\mathbb{G} : \mathcal{Z} \subset \mathbb{R}^m_{\geq 0} \to C^2$ in some sense. For this, we only want to use a finite number of variance swaps and this finite selection must be "stable" over some small time horizon.

[10] For a representation result with countably many martingales see Protter [P04] theorem 44 on page 189.

For a fixed maturity T, the value of the variance swap with maturity T at time t given a state Z_t is

$$V_t(T) = A_t + \mathbb{G}(Z_t; T - t)$$

where we define A as the running realized variance,

$$A_t := V_t(t) = \int_0^t \zeta_s \, ds \ .$$

For fixed $0 < x_1 < \cdots < x_\ell$, define further the function $\mathbb{G}|_{x_1,\ldots,x_\ell} : \mathcal{Z} \subseteq \mathbb{R}^m_{\geq 0} \to \mathbb{R}^\ell_{\geq 0}$

$$\mathbb{G}|_{x_1,\ldots,x_\ell}(z) := \Big(\mathbb{G}(z; x_1), \ldots, \mathbb{G}(z; x_\ell) \Big) \ .$$

Definition 4.17 (τ-Invertibility) *We say that the pair G (or $\mathbb{G}$) is* τ-invertible *if there exists $\ell \in \mathbb{N}$, $\tau > 0$ and $\tau < x_1 < \cdots < x_\ell$ such that*

$$\mathbb{G}|_{x_1 - t,\ldots,x_\ell - t} : \mathcal{Z} \subset \mathbb{R}^m_{\geq 0} \longrightarrow \mathbb{R}^\ell_{\geq 0}$$

is injective for all $t \in [0, \tau]$.

The idea of τ-invertibility is that is allows us to recover the value of Z on an interval $I_k := [k\tau, (k+1)\tau)$ for $k = 0, 1, \ldots$ with the stock, its realized variance process and a fixed set of variance swaps. This, in turn, can be then iterated which shows that we can actually always recover $Z = (Z_t)_{0 \leq t \leq T}$ given the stock, a finite set of variance swaps and the running realized variance. This is the subject of the following proposition:

Proposition 4.18 *Assume that (G, Z) is τ-invertible. For all $T < \infty$, there exist a finite number of variance swap maturities $T_1, \ldots, T_M$ and a function*

$$\Gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}^M_{\geq 0} \times \mathbb{R}_{\geq 0} \longrightarrow \mathcal{Z}$$

which is λ-almost surely $C^{1,2,2}$ such that

$$Z_t = \Gamma\Big(\ t\,;\ V_t(T_1)\,, \ldots\,,\ V_t(T_M)\,;\ A_t\ \Big) \tag{4.16}$$

for $t \in [0, T]$.

Consequently, the vector $(S, \hat{V}, A)$ with $\hat{V} := (V(T_1), \ldots, V(T_M))$ is Markov.

Proof – Define on each interval $I_k := [k\tau, (k+1)\tau)$ for $k = 0, 1, \ldots$ and $\ell = 0, 1, \ldots$, the variance swap maturities

$$T^k_1 := k\tau + x_1\,,\ \ldots\,,\ T^k_\ell := k\tau + x_\ell\ .$$

Due to τ-invertibility, we can recover Z on I_k via

$$Z_t = \mathbb{G}|^{-1}_{T^k_1 - t,\ldots,T^k_\ell - t}\Big(\ V_t(T^k_1) - A_t\,,\ \ldots\,,\ V_t(T^k_\ell) - A_t\ \Big) \quad t \in I_k\ ,$$

where $\mathbb{G}|^{-1}_{x_1 - t,\ldots,x_\ell - t}$ is C^2 by the inverse function theorem.

Fix now some finite T and let $K \in \mathbb{N}$ such that $K\tau \leq T < (K+1)\tau$. We can then define $M := K\ell$ variance swap maturities $T_1 := T^1_1, \ldots, T_\ell := T^1_\ell, T_{\ell+1} := T^2_1, \ldots, T_M := T^K_\ell$ (which are not necessarily distinct) and the function

$$\Gamma(t; v_1, \ldots, v_M; a) := \sum_{k=0}^K 1_{t \in I_k}\, \mathbb{G}|^{-1}_{T^k_1 - t,\ldots,T^k_\ell - t}\Big(\ v_{k\ell+1} - a\,,\ \ldots\,,\ v_{k\ell+\ell} - a\ \Big)$$

such that Z_t can be recovered on $[0, T]$ as required in (4.16). Also note that Γ is C^2 in v, a and piece-wise C^1, i.e. λ-almost surely differentiable in t. □

We are now in the position to prove the central completeness result for Markov variance swap curve market models.

THEOREM 4.19 (Complete Markov Variance Curve Market Models) *Assume that the vector* (S, Z, A) *weakly preserves smoothness and that* $\mathbb{G}$ *is* τ*-invertible.*

Then, the market $\mathbb{F}^{S,V,A}$ *is complete with respect to* (S, V)*.*

To prove this theorem, we apply very similar ideas as in the proof for theorem 4.4. In particular, the proof of the following lemma is nearly identical to the proof of lemma 4.8.

LEMMA 4.20 *For all smooth functions* $\mathbb{H} \in \mathbb{C}_K^\infty$*, the value process* $H_t := \mathbb{E}\,[\,\mathbb{H}(S_T, Z_T, A_T)\,|\,\mathcal{F}_t\,]$ *is given as a function*

$$H_t = h(t; S_t, Z_t, A_t)$$

which is C^1 *in* S *and in* Z^k *for all those* $k \in \{1, \ldots, m\}$ *for which* $\langle Z^k \rangle_T \neq 0$*. W.l.g. these are the first* m_0 *components of the vector* Z*. We then have*

$$\mathbb{H}(S_T, Z_T, A_T) = H_0 + \int_0^T \Delta_t \, dS_t + \sum_{k=1}^{m_0} \int_0^T \nu_t^k \, d\tilde{Z}_t$$

where

$$\tilde{Z}_t := Z_t - \tilde{A}_t \quad \text{and} \quad \tilde{A} := \int_0^t \mu(Z_t)\, dt \ .$$

The constant H_0 *is given as* $H_0 := \mathbb{E}\,[\,\mathbb{H}(S_T, Z_T, A_T)\,]$ *and the* $L^{loc}(S, \tilde{Z})$*-hedging ratios are*

$$\Delta_t := \partial_S h(t; S_t, Z_t, A_t) \quad \text{and} \quad \nu_t^k := \partial_{z^k} h(t; S_t, Z_t, A_t)$$

for $k = 1, \ldots, m_0$*.*

As a consequence, all $H_T \in L_+^1(\mathcal{F}^{S,V,A})$ *can be written as*

$$H_T = H_0 + \int_0^T \Delta_t \, dS_t + \sum_{k=1}^{m_0} \int_0^T \nu_t^k \, d\tilde{Z}_t \tag{4.17}$$

for $(\Delta, \nu) \in L_T^{loc}(S, \tilde{Z}; \mathbb{F}^{S,Z})$*.*[11]

Proof – Let $\mathbb{H} \in \mathbb{C}_K^\infty$. Then, the representation $H_t = h(t; S_t, Z_t, A_t) = \mathbb{E}\,[\,\mathbb{H}(S_T, Z_T, A_T)\,|\,\mathcal{F}_t\,]$ such that h is C^1 in S and for all Z^k where $\mathbb{P}[\langle Z^k \rangle_T > 0] > 0$ as stated in the lemma is just the weak preservation of smoothness property. Using the same techniques as in lemma 4.8 and the fact that $(H_t)_t$ is a martingale allows us to write

$$\begin{aligned}
\mathbb{H}(S_T, Z_T, A_T) &= H_0 + \int_0^T \Delta_t \, dS_t + \sum_{k=1}^{m_0} \int_0^T \nu_t^k \sum_{j=1}^{d} \sigma_k^j(Z_t)\, dW_t^j \\
&= H_0 + \int_0^T \Delta_t \, dS_t + \sum_{k=1}^{m_0} \int_0^T \nu_t^k \, d\tilde{Z}_t^k
\end{aligned}$$

[11] Note that the set of integrands for $(S, \tilde{Z})$ in the representation (4.17) are predictable with respect to $\mathbb{F}^{S,Z}$, not necessarily with respect to the smaller filtration $\mathbb{F}^{S,\tilde{Z}}$.

with Δ and ν as stated in the lemma. Now we apply the same ideas as for lemmata 4.9 to 4.12 to show that all $H_T \in L^1_+(\mathbb{F}^{S,Z})$ can be represented in the form (4.17) (Note that we do not prove that $\mathbb{F}^{S,\tilde{Z}}$ is equal to $\mathbb{F}^{S,Z}$ or $\mathbb{F}^{S,V,A}$. Although the process $(S,\tilde{Z})$ is extremal on $\mathbb{F}^{S,Z}$ it does not necessarily generate this filtration.)

Finally, note that $\mathbb{F}^{S,V,A} \subseteq \mathbb{F}^{S,Z}$ since $V_t(T) = G(Z_t, T-t) + \int_0^t G(Z_s;0)\,ds$. $\square$

We have therefore shown that we can replicate any payoff using S and $\tilde{Z}$. Since the latter is not tradable, we need to replicate it by itself in order to prove theorem 4.19.

Proof of theorem 4.19– Let $H_T \in L^1_+(\mathcal{F}^{S,V,A}_T)$ such that, according to lemma 4.20,

$$H_T = H_0 + \int_0^T \Delta^S_t \, dS_t + \sum_{k=1}^{m_0} \int_0^T \Delta^k_t \, d\tilde{Z}^k_t$$

for $(\Delta,\nu) \in L^{\text{loc}}_T(S,\tilde{Z};\mathbb{F}^{S,Z})$. Note that because of τ-invertibility, $\mathbb{F}^{S,Z} = \mathbb{F}^{S,V,A}$, hence we can assume w.l.g. that $(\Delta,\nu) \in L^{\text{loc}}_T(S,\tilde{Z};\mathbb{F}^{S,V,A})$.

Let now Γ and $T_1,\dots,T_M$ be as in proposition 4.18. Then,

$$Z_t = \Gamma\left(t; V_t(T_1),\dots,V_t(T_M); A_t\right) \ ,$$

and Itô's formula given the fact that $\tilde{Z}$ is a local martingale shows that

$$d\tilde{Z}_t = \sum_{\ell=1}^{M} \partial_{v^\ell}\Gamma\left(\cdots\right)\, dV_t(T_\ell) \ ,$$

hence the market $\mathbb{F}^{S,V,A}$ is complete. $\square$

The following corollary identifies the form of the hedging ratios for the cases where the value process is given as a sufficiently differentiable function of S and the relevant variance swaps. Not surprisingly, the hedging ratios are just the derivatives with respect to the spot prices of the market instruments.

Corollary 4.21 (Delta-Hedging with Variance Swaps) *Assume that (S,Z,A) weakly preserve smoothness and that $\mathbb{G}$ is τ-invertible. Moreover, assume that $H_T \in L^1_+(\mathcal{F}^{S,V}_T)$ is a payoff such that its price process $H = (H_t)_{t\in[0,T]}$ can be written as*

$$H_t = h(t; S_t, Z_t, A_t)$$

for some value function h which is C^1 in S and Z.

Then, there exists a function $\hat{h}$ such that

$$H_t = \hat{h}\Big(t; S_t, V_t(T_1),\dots,V_t(T_M); A_t\Big) \tag{4.18}$$

where $T_1,\dots,T_M$ are the maturities from proposition 4.18. Moreover, h_t is smooth in S and the V-arguments, such that H_T can be hedged using

$$dH_t = \partial_S \hat{h}(\cdots)\, dS_t + \sum_{k=1}^{M} \partial_{V_k}\hat{h}(\cdots)\, dV_t(T_k) \ . \tag{4.19}$$

This corollary covers both standard European options on the stock and options on variance as defined above.

Proof – Following the proof of theorem 4.19, we simply have $\hat{h} := h \circ \Gamma$. □

NOTATION 1 *From now on, we will write the value function of a process H_t in terms of S and the states Z as "h", while we use a hat, "$\hat{h}$", for the respective function which is parameterized by S and the variance swaps $V(T_1), \dots, V(T_M)$.*

COROLLARY 4.22 *Given the states (S, Z), the hedging ratios given in corollary 4.21 can be computed as follows: first, compute the $\mathbb{R}^{m\times M}$-matrix of coefficients*

$$\Psi_t(Z_t) = \Big(\partial_{z_i}\mathbb{G}(Z_t, T_k - t)\Big)_{i=1,\dots,m;k=1,\dots,M}$$

for which by assumption

$$d\begin{pmatrix} V_t(T_1) \\ \vdots \\ V_t(T_M) \end{pmatrix} = \Psi_t(Z_t)\, d\begin{pmatrix} \tilde{Z}_t^1 \\ \vdots \\ \tilde{Z}_t^m \end{pmatrix}$$

holds. Since $\Psi_t(Z_t)$ has full rank m, we find a generalized inverse $\Psi_t(Z_t)^{-1} \in \mathbb{R}^{M\times m}$ and can define the row vector

$$\begin{pmatrix} \nu_t^1 \\ \vdots \\ \nu_t^M \end{pmatrix}' := \begin{pmatrix} \partial_{z_1} h(t, V_t(t); S_t, Z_t) \\ \vdots \\ \partial_{z_m} h(t, V_t(t); S_t, Z_t) \end{pmatrix}' \Psi_t(Z_t)^{-1} \ . \tag{4.20}$$

Also let

$$\Delta_t := \partial_S h(t, V_t(t); S_t, Z_t) \ .$$

Then, H is replicated by

$$H_T = H_0 + \int_0^T \Delta_t(t, V_t(t); S_t, Z_t)\, dS_t + \sum_{k=1}^M \int_0^T \nu^k(t, V_t(t); S_t, Z_t)\, dV_t(T_k)$$

with slight abuse of notation.

In the context of chapter 5 below, note that computing (4.20) is equivalent to "neutralizing the risk" with respect to the states: indeed, assume that $\Psi_t(Z_t)$ and the row vector $\psi_t := (\partial_{z^1} h(t, V_t(t); S_t, Z_t), \dots, \partial_{z^m} h(t, V_t(t); S_t, Z_t))$ are known.

Then,

$$\min_{v\in\mathbb{R}^M} \| \ {\psi_t}' - v'\Psi_t(Z_t) \ \|_2 \tag{4.21}$$

is minimized by some row-vector v if and only if $v' = \partial_z h' \Psi_t(Z_t)^{-1}$ for a generalized inverse of the matrix $\Psi_t(Z_t)$.

See also the discussion on "parameter-hedging" in chapter 5.

4.2.4 Hedging in classic Stochastic Volatility Models

In this short subsection, we want to show that in sufficiently well-behaved one-factor models, we can hedge options on variance using the variance swap with the same maturity as the option.

COROLLARY 4.23 *Assume that $Z = (\zeta, A)$ solves*

$$\left.\begin{array}{rcl} d\zeta_t &=& \mu(\zeta_t)\,dt + \sigma(\zeta_t)\,dW_t^1 \\ dA_t &=& \zeta_t\,dt \end{array}\right\}$$

for differentiable μ and σ with locally Lipschitz derivatives.
Then,

(a) The process Z weakly preserves smoothness.

(b) The variance swap curve functional of the model, $\mathbb{G}(z;x) := \mathbb{E}\left[\, A_x \mid \zeta_0 = z \,\right]$, is globally invertible.

Moreover, we can replicate any payoff $H_T \in L^1_+(\mathcal{F}^Z)$ with the variance swap with the same maturity T as the option, i.e. there is some $\nu \in L^{loc}_T(V(T))$ such that

$$H_T = \mathbb{E}\left[\, H_T \,\right] + \int_0^T \nu_t\, dV_t(T)\ . \tag{4.22}$$

As before, we will denote by ζ^z the process ζ started in z.

Proof – First, (ζ, A) weakly preserves smoothness following proposition 4.5. As a next step, we prove that

$$z \mapsto \mathbb{G}(z;x) := \mathbb{E}\left[\int_0^x \zeta_t^z\, dt\right]$$

is C^1, strictly increasing and therefore invertible for all $x > 0$.

Indeed, differentiability follows because $z \mapsto \zeta_t^z$ is differentiable with derivative $\zeta^{(z,1)}$ (see equation (4.5) above), which in turn is a supermartingale. Hence,

$$\partial_z \mathbb{E}\left[\int_0^x \zeta_t^z\, dt\right] = \mathbb{E}\left[\int_0^x \zeta_t^{(z,1)}\, dt\right]\ .$$

It remains to prove that $z \mapsto \mathbb{E}\left[\int_0^x \zeta_t^z\, dt\right]$ is strictly increasing. To this end, let $z < y$ and let $\tau := \inf\{t : \zeta_t^z \geq \zeta_t^y\}$. By continuity of $z \mapsto \zeta_t^z$, we have $\tau > 0$ and $\zeta_0^z < \zeta_0^y$. Moreover, $\zeta_t^z = \zeta_t^y$ for all $t > \tau$. Consequently, $z \mapsto \int_0^x \zeta_t^z\, dt$ is a strictly increasing function.

Theorem 4.4 yields therefore that for all $\epsilon > 0$, the market $L^1_+(\mathcal{F}^V_{T-\epsilon})$ is complete with respect to $(V(T), A)$. It remains to show that this is also true for $\epsilon = 0$, but this follows since we only have to show the existence of a hedging strategy along a localizing sequence of stopping times, which we choose as $\tau_n := T - 1/n$: indeed,

$$H_t^n := \mathbb{E}\left[\, H_T \mid \mathcal{F}_{\tau_n \wedge t} \,\right] = \mathbb{E}\left[\, H_T \mid \zeta_{\tau_n \wedge t}, A_{\tau_n \wedge t} \,\right]$$

can be replicated for all n. This yields the representation (4.22) in $L^{\mathrm{loc}}_T(V(T))$. $\square$

Chapter 5

Hedging in Practice

The previous chapter showed how we can use variance curve models to hedge exotic products in theory: we first determine the model of choice, then we evaluate the payoff and finally we compute the corresponding hedging ratios with respect to stock and reference instruments (the latter being variance swaps in our models).

According to the remarks following corollary 4.22, these hedging ratios can be computed by a simple procedure: in addition to the target product, choose a number of reference instruments such that the sensitivities of the overall portfolio with respect to the states vanish.

In practise, however, there is another task involved: the issue of selecting appropriate *parameters* for the model and to protect the portfolio against changes of these parameters.

This chapter is thus devoted to this "parameter-hedging". We start in the first section 5.1 below by introducing the concepts of "models", "calibration" and "meta-models" before we explain this idea. This is based on ideas from [B06b].

The following section 5.2 will then discuss how this approach can be implemented: we describe an effective algorithm which allows the selection of a cheapest portfolio of liquid options such that the sensitivity of the joint position of exotic payoff and portfolio to changes in states and parameters of the model is kept within a specified tolerance. The algorithm allows imposition of a range of reasonable market constraints such as limits on transaction sizes and overall model error (discrepancy between market prices and model values of the liquid instruments).

In the final section 5.3, we will then show the theoretical result that meta-models based on Heston's model (example 3.3), the exponential-OU model (example 3.8) or the double mean-reverting models (which are discussed in length in part III of the thesis) are not free of "dynamic arbitrage" if the speeds of mean-reversion are not kept constant: the value processes which are the result of instant recalibration are not local martingales. Moreover we will also introduce *entropy swaps*, which we will use to show that in the particular case of Heston's model, the product of correlation and "volatility of variance" must also be kept constant.

Appendix A.1.2 discusses practical applications of entropy swaps and a related instrument, *gamma swaps*.

5.1 Model and Market

To illustrate the ideas discussed in the later sections of this chapter, we start with a guiding example. To this end, consider Heston's model [H93] which is defined on a stochastic base

$\bar{\mathbb{W}} = (\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ with $\bar{\mathbb{F}}$-extremal two-dimensional Brownian motion $\bar{W}$ as the solution to

$$\left.\begin{array}{rcl} d\bar{\zeta}_\tau &=& \kappa(\theta - \bar{\zeta}_\tau)\,dt + \nu\sqrt{\bar{\zeta}_t}\,d\bar{W}^1_\tau \qquad \bar{\zeta}_0 = \zeta_0 \\ d\bar{X}_t &=& \sqrt{\bar{\zeta}_\tau}\,d(\rho\bar{W}^1_\tau + \sqrt{1-\rho^2}\bar{W}^2_\tau) \\ \bar{S}_\tau &=& S_0\,\mathcal{E}_\tau(\bar{X})\ . \end{array}\right\} \tag{5.1}$$

This model has the states $(\bar{S}_0, \bar{\zeta}_0) \in \mathbb{R}^+ \times \mathbb{R}^+_0$ and the parameters $\chi = (\kappa, \theta, \nu, \rho) \in \Upsilon$ with $\Upsilon := R^{+3} \times (-1, +1)$; the constant κ is called the "speed of mean-reversion", θ is the "level of mean-reversion", ν is the "volatility of variance" (or "VolOfVol") and ρ is called "correlation".

We will often use superscripts to indicate the states and parameters of a model: for example, $\bar{S}^{S_0,\zeta_0;\chi}_\tau(\bar{\omega})$ is the value of the stock in the model given the path $\bar{\omega} \in \bar{\Omega}$ where $\bar{S}_0 := S_0$ and $\bar{\zeta}_0 := \zeta_0$. We will also use $\xi := (\zeta_0, \chi)$ to denote the *configuration* of the model.

However, this is just the model. Assume now that the real stock price process is a strictly positive martingale $S = (S_t)_{t\geq 0}$ defined on a different stochastic base $\mathbb{W} = (\Omega, \mathcal{F}_\infty, \mathbb{F}, \mathbb{P})$ which supports some finite-dimensional extremal Brownian motion W.

To introduce a few ideas, we also assume that a single variance swap with maturity T is traded. Its price process in the market world is denoted by $V(T) = (V_t(T))_{t\leq T}$. In section 5.1.1 we will consider a richer market where a range of liquid options is traded.

Pricing

Assume we want to use Heston's model above to evaluate at "market time" $t = 0$ some illiquid payoff, for example a call:

$$\left(S_{T_1} - K\right)^+ \tag{5.2}$$

where $T_1 \leq T$.

This requires us to specify the state ζ_0 and the parameters χ of the Heston model. For the moment, note that the price of the liquid variance swap in Heston's model is given as

$$\breve{V}_0(T) = \mathbb{G}(\zeta_0, \chi; T) := \theta T + \left(\bar{\zeta}_0 - \theta\right)\frac{1 - e^{-\kappa T}}{\kappa}\ . \tag{5.3}$$

This means that once some parameters χ are specified, we can invert this function given the market price $V_0(T)$ to obtain ζ_0. Let us therefore assume that we have indeed chosen a suitable configuration $\xi = (\zeta_0, \chi)$ (we discuss below how this is usually done). Then, we can compute

$$\breve{H}_0 := h^\chi(0; S_0; \zeta_0) := \bar{\mathbb{E}}\left[\left(\bar{S}^{\zeta_0,\chi}_{T_1} - K\right)^+ \,\middle|\, \bar{S}_0 = S_0, \bar{\zeta}_0 = \zeta_0\right]\ .$$

We will regard this as the *value* of the payoff (5.2) given by our model and the configuration $\xi = (\zeta_0, \chi)$. We do not call it *price*, because it does not necessarily reflect the cost of a replication strategy in the real world.

Initial Hedging

According to the logic inherent in Heston's model (cf. chapter 4), we should hedge our exposure to the risk arising from both a change in S and in ζ. Corollary 4.22 shows that this can be done by using the additional variance swap $V(T)$: first, compute the model-sensitivities of $\breve{H}_0$ with respect to S and ζ,

$$\breve{\Delta}^C_0 := \partial_S h^\chi(0; S_0; \zeta_0) \quad \text{and} \quad \breve{\psi}^C_0 := \partial_\zeta h^\chi(0; S_0, \zeta_0)\ .$$

Now do the same with the model-price of the liquid variance swap (5.3). We obtain

$$\breve{\Delta}_0^V := \partial_S \mathbb{G}(\zeta_0, \chi; T) = 0 \quad \text{and} \quad \breve{\psi}_0^V := \partial_\zeta \mathbb{G}(\zeta_0, \chi; T) \ .$$

Hence, if we are short one call (5.2), we can hedge our overall exposure to (S, ζ) by buying $\breve{\Delta}_0^C$ shares and $\breve{\psi}_0^C / \breve{\psi}_0^V$ variance swaps (cf. corollary 4.22). Our overall value of the portfolio in the model world is then

$$-\breve{H}_0 + \breve{\Delta}_0 S_0 + \breve{\nu}_0 \breve{V}_0(T) \tag{5.4}$$

with

$$\breve{\Delta}_0 := \breve{\Delta}_0^C \quad \text{and} \quad \breve{\nu}_0 := \frac{\breve{\psi}_0^C}{\breve{\psi}_0^V} \ . \tag{5.5}$$

Instantaneous Hedging

The above portfolio (5.4) is insensitive with respect to moves in S and ζ over short time-periods *in the Heston model*. Assume now that at some later *market time* $t > 0$, we want to rebalance our hedge (note that we use τ to denote model time and t to denote market time).

Since the stock and the liquid variance swap are liquidly traded, we observe the historical path $\mathbb{S}^t(\omega) = S_{u:u \le t}(\omega) \in C[0,t]$ of S and the path $V_{u:u\le t}(T)(\omega)$ of the liquid variance swap. We now want to compute the value of the call.

Assuming we are confident in our initial parameter choice χ, we have to determine a sensible value for the short variance $\zeta_t(\omega)$ in the real-world. As before, we can imply $\zeta_t(\omega)$ using (5.3) and the historic observation $\langle \log \mathbb{S}^t(\omega) \rangle$ (i.e. the value of the variance swap which just matured).

The new call value is given by

$$\breve{H}_t(\omega) := h^\chi\left(t; S_t(\omega), \zeta_t(\omega)\right) := \bar{\mathbb{E}}\left[\left(\bar{S}_{T_1-t}^{\zeta_t(\omega),\chi} - K \right)^+ \,\middle|\, \bar{S}_0 = S_t(\omega), \bar{\zeta}_0 = \zeta_t(\omega) \right] \ .$$

Note that we have in fact evaluated a new, "shifted" payoff

$$\left(\bar{S}_{T_1 - t}(\bar{\omega}) - K \right)^+ \tag{5.6}$$

conditionally on $\bar{S}_0 = S_t(\omega)$ and $\bar{\zeta}_0 = \zeta_t(\omega)$. This concept of "shifting the payoff" will be formalized below.

The hedge ratios $\breve{\Delta}$ and $\breve{\nu}$ can now be computed as before (5.5) instantaneously for every time t. We obtain a portfolio value process in the market world which follows

$$\breve{P}_t(\omega) := \breve{H}_0 + \int_0^t \breve{\Delta}_t(\omega)\, dS_t(\omega) + \int_0^t \breve{\nu}_t(\omega)\, dV_t(\omega) \ ,$$

but in general

$$\breve{P}_t(\omega) \ne \breve{H}_t(\omega) \ .$$

We can write this as

$$\breve{H}_t(\omega) = \breve{H}_0 + \int_0^t \breve{\Delta}_t(\omega)\, dS_t(\omega) + \int_0^t \breve{\nu}_t(\omega)\, dV_t(\omega) + \Gamma_t(\omega)$$

where Γ is the *profit/loss process*.

It is clear that if the market behaves like a Heston model with the parameter χ, then $\Gamma \equiv 0$. In general, however, Γ introduces swings in our profit/loss computation.[1] This means that we can not expect to hedge our exotic products perfectly. Nonetheless, the above procedure is a natural approach if we do not know the real market dynamics. But we have omitted one crucial detail: the determination of the parameter vector χ.

5.1.1 Additional Market Instruments

In the previous section, we have assumed that only the stock S and a single variance swap were traded. For a fixed parameter vector χ of the Heston model, this implied that there is an unique short variance ζ_0 such that the model is consistent with the observed market data. However, in real life markets, a full range of liquid options is usually traded.

Let us therefore fix some $T > 0$ and assume that in addition to S, there are d liquid options with price processes $C^1, \dots, C^d$ quoted. These price processes are assumed to have terminal payoffs

$$C_T^\ell(\omega) = \mathbb{C}^\ell\Big(S_{u:u\le T}(\omega)\Big)$$

given in terms of Borel *payoff functions*

$$\mathbb{C}^\ell : C_+[0,T] \longrightarrow \mathbb{R}_0^+$$

for $\ell = 1, \dots, d$.[2]

DEFINITION 5.1 *We define the set of all* admissible payoff functions *as*

$$\mathcal{X} := \Big\{ \ \mathbb{H} : C_+[0,T] \longrightarrow \mathbb{R}_0^+ \ \textit{is Borel} \ \Big\} \ .$$

ASSUMPTION 5 *We assume that all traded payoffs are integrable for all configurations of our model, i.e. that*

$$\mathbb{C}^\ell\left(\bar{S}^{S_0,\zeta_0,\chi}_{u:u\le T}\right) \in L^1_+\left(\bar{\mathbb{P}}^{S_0,\xi}\right)$$

for all $\ell = 1, \dots, d$, $S_0 \in \mathbb{R}^+$ and $\xi = (\zeta_0, \chi) \in \Xi := R_0^+ \times \mathcal{X}$.

This is a very natural assumption: if we are trading in a market with liquid instruments $C^1, \dots, C^d$ which have finite prices, a model which assigns an infinite price to one of these instruments is clearly not a reasonable candidate to price and hedge exotic products.

EXAMPLE 5.2 *The payoff function $\mathbb{H}$ of a variance swap with maturity $T_1 \le T$ is given as*

$$\mathbb{H}(f) := \langle \log f \rangle_{T_1} := \lim_{n\uparrow\infty} \sum_{i=1}^{n} \left(\log \frac{f(t_i)}{f(t_{i-1})} \right)^2 \tag{5.7}$$

where $\tau = (\tau_n)_{n\in\mathbb{N}}$ with $\tau_n = (0 = t_1^n < \dots < t_n^n = T_1)$ is a refining partition of $[0,T_1]$, i.e. $\lim_{n\uparrow\infty} \sup_{i=1,\dots,n} |t_i^n - t_{i-1}^n| = 0$.

[1] In appendix C.1 we show how the profit/loss process Γ can be computed in a simple stochastic volatility framework.

[2] We used $C_+[0,T]$ to denote the set of all right-continuous non-negative functions $f : [0,T] \to \mathbb{R}_0^+$.

REMARK 5.3 *Definition 5.1 only allows payoff functions which depend directly on the path of the underlying stock price. In general, this excludes, for example, options on variance swaps.*[3]

We have limited ourselves in this chapter to the above definition as a matter of convenience; in chapter 7 we will consider payoff functions which depend on the paths of finitely many observable liquid instruments.

Shifting the Payoff

For all $\mathbb{H} \in \mathcal{X}$ and all configurations ξ, we now define the initial value of $\mathbb{H}$ as

$$U(S_0, \xi; \mathbb{H}) := \bar{\mathbb{E}}\left[\mathbb{H}\left(\bar{S}^{S_0,\zeta_0,\chi}_{u:u\le T}\right) \,\middle|\, \bar{S}_0 = S_0, \bar{\zeta}_0 = \zeta_0 \right]$$

(which might be infinite). At some later time $t > 0$ the stock price has already moved along a path $\mathbb{S}^t(\omega) := S_{u:u\le t}(\omega) \in C_+[0,t]$, we therefore need to generalize the shifting of the payoff which we introduced above (5.6). To this end, we define the "gluing operators"

$$\theta^\omega_t : f \in C_+[0,T] \longmapsto \theta_t(f) \in C_+[0,T]$$

by

$$\theta^\omega_t(f)(x) := \mathbb{S}^t(\omega)(x) 1_{x<t} + f(x-t) 1_{x\ge t}$$

for $x \in [0,T]$. The operator θ^ω_t "glues" f right-continuously at the end of $\mathbb{S}^t(\omega)$ (note that the values of f past $T-t$ are ignored).

The idea is to use θ^ω_t to shift the realized path $\mathbb{S}^t(\omega)$ in front of the model stock price $\bar{S} = (\bar{S}_\tau)_{\tau\ge 0}$: indeed, we have

$$\theta^\omega_t\left(\bar{S}_{u:u\le T}(\bar{\omega})\right)(x) = S_x(\omega) 1_{x<t} + \bar{S}_{x-t}(\bar{\omega}) \tag{5.8}$$

for $\bar{\omega} \in \bar{\Omega}$.

We can therefore use our abstract concept of a payoff $\mathbb{H}$ as an element of $\mathcal{X}$ by defining the *shifted payoff function* $\mathbb{H}_t(\omega) \in \mathcal{X}$ via

$$\mathbb{H}_t(\omega) := \mathbb{H} \circ \theta^\omega_t : C_+[0,T] \longrightarrow \mathbb{R}^+_0 \ .$$

This shifted payoff allows us to compute the model-price of the payoff function $\mathbb{H}$ given the configuration $\xi = (\zeta, \chi)$ at some time $t > 0$ as

$$U(S_t(\omega), \xi; \mathbb{H}_t(\omega)) \quad = \quad \bar{\mathbb{E}}\left[\mathbb{H}^\omega_t\left(\bar{S}^{S_0,\zeta,\chi}_{u:u\le T}\right) \,\middle|\, \bar{S}_t(\omega) = S_0, \bar{\zeta}_0 = \zeta \right] \ .$$

EXAMPLE 5.4 *Let us again consider the case of a variance swap with maturity T_1 which has the payoff function $\mathbb{H}(f) := \langle \log f \rangle_{T_1}$ defined in (5.7).*

(a) A time $t = 0$ and given some $\xi_0 = (\zeta_0, \chi)$, we simply have

$$U(S_0, \xi_0; \mathbb{H}) = \bar{\mathbb{E}}\left[\int_0^{T_1} \bar{\zeta}^{\zeta_0,\chi}_\tau \, d\tau \right] = \mathbb{G}(\zeta_0, \chi; T_1)$$

where $\mathbb{G}$ is defined in (5.3).

[3] In contrast to realized variance, the price of a variance swap is not measurable with respect to the filtration generated by the stock price unless the stock price itself already constitutes a complete market.

(b) *At some later time $t \in (0, T_1)$, assume we have determined a new short variance $\zeta_t(\omega)$ and set $\xi_t(\omega) := (\zeta_t(\omega), \chi)$.*

The value of the variance swap for $t \in [0, T_1]$ is now

$$\begin{aligned} U\Big(S_t(\omega), \xi_t(\omega); \mathbb{H}_t(\omega)\Big) &= \langle \log \mathbb{S}^t(\omega)\rangle_t + \bar{\mathbb{E}}\left[\int_0^{T_1-t} \bar{\zeta}_\tau^{\zeta_t(\omega),\chi}\, d\tau\right] \\ &= \langle \log \mathbb{S}^t(\omega)\rangle_t + \mathbb{G}\Big(\zeta_t(\omega), \chi; T_1 - t\Big)\ . \end{aligned}$$

This is the actual realized variance of S up t in the market, $\langle \log \mathbb{S}^t(\omega)\rangle_t$, plus the remaining expected remaining variance up to T_1 in the model given a short variance of $\bar{\zeta}_0 := \zeta_t(\omega)$.

(c) *At maturity T_1, the value of the payoff is just*

$$U\Big(S_{T_1}(\omega), \xi_{T_1}(\omega); \mathbb{H}_{T_1}(\omega)\Big) = \langle \log \mathbb{S}^{T_1}(\omega)\rangle_{T_1}\ ,$$

i.e. indeed the realized variance of the stock.

Calibration and Recalibration

At this point, we can formalize the concept of a "model" beyond the example of Heston:

DEFINITION 5.5 *A* model U *is a continuous map*

$$\begin{aligned} U : \mathbb{R}^+ \times \Xi \times \mathcal{X} &\longrightarrow \mathbb{R}_0^+ \\ (S_0, \xi, \mathbb{H}) &\longmapsto U(S_0, \xi; \mathbb{H}) \end{aligned}$$

which is differentiable in S_0 and ξ (the set Ξ is assumed to be an open subset of $\mathbb{R}^K$). Each index ξ is called a configuration *of the model.*

REMARK 5.6 *Commonly, we assume that $U(S_0, \xi; \cdot)$ is a* price system, *i.e.*

(a) *It is linear.*

(b) *It is positive.*

(c) *$U(S_0, \xi; c) = c$ for all constants $c \in \mathbb{R}_0^+$.*

We do not need these assumptions here. See, however, Föllmer/Schied [FS04] for more details.

With definition 5.5 at hand and with the example of Heston's model in mind, we now assume again that we want to evaluate a payoff $\mathbb{H} \in \mathcal{X}$. In contrast to the initial example in the previous section, we now need to determine a configuration vector $\xi_t(\omega) \in \Xi$ at all times t from the traded market prices $C^1, \dots, C^d$. To this end, define the vector

$$U(S_t(\omega), \xi; \mathbb{C}_t(\omega)) := \Big(\ U(S_t(\omega), \xi; \mathbb{C}_t^1(\omega)), \dots, U(S_t(\omega), \xi; \mathbb{C}_t^d(\omega))\ \Big)\ .$$

for all $\xi \in \Xi$.

A natural configuration at time t is the *calibrated* configuration vector

$$\xi_t^*(\omega) := \mathrm{argmin}_{\xi\in\Xi} \left\|\ C_t(\omega) - U(S_t(\omega), \xi; \mathbb{C}_t(\omega))\right\|^*, \tag{5.9}$$

where $\|\cdot\|^*$ is a suitable norm.

The program (5.9) formalizes the idea of using the observed market data to imply optimal states and parameters of a model (we assume that the algorithm which solves (5.9) yields a unique solution).[4] If this is exercised instantaneously at every time t, the above calibration program will yield an $\mathbb{F}$-adapted configuration process $\xi^* = (\xi^*_t)_{t\in[0,T]}$ which can be used to compute the "instantaneously recalibrated" value process

$$\breve{H}^*_t(\omega) := U\Big(S_t(\omega), \xi^*_t(\omega); \mathbb{H}_t(\omega)\Big)$$

of the exotic payoff $\mathbb{H}$. However, the user may not wish to recalibrate the model too frequently (in the real world, truly instantaneous hedging is infeasible). Indeed, any $\mathbb{F}$-adapted process $\xi = (\xi_t)_{t\in[0,T]}$ can be used to define a value process for the exotic payoff $\mathbb{H}$. The resulting map between payoffs and value processes is what we want to call the "meta-model" of the institution:

Definition 5.7 *The* meta-model of the institution *based on the model U and the configuration process ξ is the map*

$$\mathcal{U} : (t, \mathbb{H}) \longmapsto U\Big(S_t, \xi_t; \mathbb{H}_t\Big) \ .$$

From corollary 4.22 we deduce:

Proposition 5.8 *If the market dynamics are given as a particular configuration ξ_0 of the model, and if the market $L^1(\mathcal{F}^S_T)$ is complete with respect to $(S, C^1, \ldots, C^d)$ for the parameter vector χ, then*

$$\breve{H}_t(\omega) := U(S_t(\omega), \xi_t(\omega); \mathbb{H}_t(\omega)) \ .$$

with ξ defined by (5.9) for an $\|\cdot\|_2$-equivalent norm $\|\cdot\|^*$ is the unique fair price process of $\mathbb{H}$ in the market.

In general, however, this does not hold. While the underlying model such as Heston's is usually free of arbitrage in itself,[5] the value process generated by the meta-model may well exhibit "dynamic arbitrage" in the following sense:

Definition 5.9 (Dynamic Arbitrage) *We say $\mathcal{U}$ is free of* dynamic arbitrage *if there exists an equivalent measure $\mathbb{Q} \approx \mathbb{P}$ under which all value processes for all admissible payoffs are local martingales.*

If the meta-model $\mathcal{U}^*$ exhibits dynamic arbitrage in the sense above, then there exists a self-financing trading strategy in admissible payoffs which has no or negative initial cost, which produces a $\mathbb{P}$-almost surely non-negative payoff and which has a non-zero $\mathbb{P}$-probability of actually being positive.

In section 5.3, we will show that a meta-model based on Heston's model or on one of the models introduced in sections 3.1 or 3.2 is not free of dynamic arbitrage if the speeds of mean-reversion are not kept constant.

[4] Even though a numerical minimization routine will usually only return a local minimum, such a numerical solution usually still depends deterministically on the input data and is therefore unique.

[5] See Föllmer/Schied [FS04] for details on price systems and absence of arbitrage.

5.2 Parameter Hedging in Practise

In practise, it is difficult to assess whether a meta-model exhibits dynamic arbitrage (indeed, it probably does), but this is not our main concern. Even if we accept that the value process $\breve{H}$ is not a true price process, the main question is how we can minimize our exposure to the risk of a change of the configuration vector.

Indeed, recall (5.5) where we computed the hedging ratios for the stock price sensitivity and the sensitivity of the payoff with respect to ζ using corollary 4.22. This approach made sense because Heston's model is complete if we hedge the risks of a payoff with respect to stock and short variance. However, we still apply it even though we do not expect the market to move exactly according to Heston's dynamics.

The heuristic idea is that if Heston's model is "close" to the true dynamics, then the hedging ratios will be reasonably good, too (appendix C.1 provides an example of the impact of a wrongly specified stochastic volatility model).

The idea of "parameter-hedging" is now to implement the same "hedge" for all elements of the configuration vector, not only the states.[6]

Parameter-Hedging

For $t \in [0,T]$, let

$$\breve{\Psi}_t := \Big(\partial_{\xi^k} U(S_t, \xi_t; \mathbb{C}_t^\ell))\Big)_{\ell=1,\dots,d;k=1,\dots,K}$$

be the $\mathbb{R}^{d\times K}$ matrix of sensitivities of the model values of the market instruments with respect to the configuration vector $\xi = (\xi^1, \dots, \xi^K)$.

Also compute the row-vector of sensitivities of the value of $\mathbb{H}$ with respect to the configuration,

$$\breve{\psi}_t := \big(\ \partial_{\xi^1} U(S_t, \xi_t; \mathbb{H}_t)), \dots, \partial_{\xi^K} U(S_t, \xi_t; \mathbb{H}_t))\ \big) \in \mathbb{R}^K \ .$$

Let now $\breve{\Psi}_t^{-1} \in \mathbb{R}^{K\times d}$ be a generalized inverse of $\breve{\Psi}_t$. Just as in corollary 4.22 for the states, define now the d-dimensional row-vector

$$\breve{\nu}_t := \breve{\psi}_t\, \breve{\Psi}_t^{-1}\ ,$$

which will as usual be a solution to

$$\mathrm{argmin}_{\nu\in\mathbb{R}^K} \Big\|\ \breve{\psi}_t' - \nu' \breve{\Psi}_t\ \Big\|_2 \tag{5.10}$$

(the introduction of a weighted norm is straight-forward). As a result, the portfolio

$$-\breve{H}_t + \sum_{\ell=1}^d \nu_t^\ell \breve{C}_t^\ell + \breve{\Delta}_t S_t$$

with

$$\breve{\Delta}_t := \partial_S \left(U(S_t), \xi_t; \mathbb{H}_t) - \sum_{\ell=1}^d \nu_t^k \partial_S U(S_t, \xi_t; \mathbb{C}_t^\ell) \right)$$

[6]In fact, the distinction between states and parameters can be blurred: what happens, for example, in Heston's model if the "volatility of variance" ν is zero and $\bar{\zeta}$ becomes a deterministic function of time? What if moreover, $\theta = \zeta_0$, in which case Heston's model degenerates to Black&Scholes' constant volatility model?

has L^2-minimal exposure to any change of the configuration and is insensitive to any change in the stock price.

We will now discuss how a relaxed version of (5.10) which also takes into account additional real-life constraints can be implemented.

5.2.1 Constrained Parameter-Hedging in Practise

Without loss of generalization, we consider problem (5.10) at time $t = 0$.

We are given a vector of market prices $C_0 = (C_0^1, \dots, C_0^d)$ of liquid options with payoffs $(\mathbb{C}^1, \dots, \mathbb{C}^d)$. Their model prices are given by $\breve{C}_0^\ell \equiv \breve{C}^\ell(\xi)$ for all $\ell = 0, \dots, d$.

We also have an admissible payoff $\mathbb{H}$ with initial value $\breve{H}_0 \equiv \breve{H}(\xi)$. The aim is to construct a portfolio of liquid options such that the combined position of $\mathbb{H}$ and this portfolio has minimal sensitivity to the configuration values ξ.

Remark 5.10 *We disregard the stock sensitivities here since we assume that all liquid instruments are traded delta-neutral, i.e. with the associated initial delta hedge.*[7]

We adopt the notation from above: The matrix of derivatives of $\breve{C}$ with respect to ξ is denoted by

$$\breve{\Psi} := \begin{pmatrix} \partial_{\xi^1}\breve{C}^1 & \cdots & \partial_{\xi^K}\breve{C}^1 \\ \vdots & \ddots & \vdots \\ \partial_{\xi^1}\breve{C}^d & \cdots & \partial_{\xi^K}\breve{C}^d \end{pmatrix} \in \mathbb{R}^{d\times K} .$$

and the vector of sensitivities of $\breve{H}$ is

$$\breve{\psi} := \left(\partial_{\xi^1}\breve{H}, \dots, \partial_{\xi^K}\breve{H} \right) \in \mathbb{R}^K .$$

Unconstrained Optimization

In case we are just interested in *some* solution to our problem, we can define as above

$$\breve{\nu}_{\text{simple}} := \breve{\psi}\, \breve{\Psi}^{-1} \in \mathbb{R}^d$$

where $\breve{\Psi}^{-1} \in \mathbb{R}^{K\times d}$ denotes a generalized inverse of $\breve{\Psi}$.[8] If $\breve{\Psi}$ has not full rank, then $\breve{\nu}_{\text{simple}}$ will be the orthogonal projection of $\breve{\psi}$ onto the spaces spanned by $\breve{\Psi}$, hence an L^2-optimal fit: the position

$$-\breve{H} + \sum_{k=1}^{d} \breve{\nu}_{\text{simple}}^k\, \breve{C}^k \tag{5.11}$$

has L^2-minimal possible exposure to changes in ξ.

However, this approach will yield quite an arbitrary hedging portfolio. Usually, we have many traded options which are available at very different costs (in the form of bid/ask spreads i.e. transaction costs) and which have very different sensitivities to ξ. It is clear that under such

[7] Otherwise, we run the same program as described here with $\hat{\xi} = (S_0, \xi)$ and $\hat{C} = (S_0, C^1, \dots, C^d)$.

[8] Consider a singular value decomposition $\breve{\Psi} = UDV^T$ for orthogonal $U \in \mathbb{R}^{K\times K}$, $V \in \mathbb{R}^{d\times d}$ and a diagonal matrix $D \in \mathbb{R}^{d\times K}$. Then,

$$\breve{\Psi}^{-1} := VD^{-1}U^T$$

is a generalized inverse of $\breve{\Psi}$ ($D^{-1} \in \mathbb{R}^{K\times d}$ is a diagonal matrix where ${D^{-1}}_i^i$ is equal to $1/D_i^i$ if $D_i^i > 0$ or zero otherwise).

circumstances an unconstrained "blind" minimization like the one above is not a very useful solution (for each subset of m options for which the sensitivity matrix has full rank we can find a corresponding position $\nu^1, \ldots, \nu^m$ such that (5.11) has zero sensitivity to the configuration vector).

Moreover, we may not require total elimination of the sensitivities of our portfolio. A certain exposure might be acceptable: assume, for example, that our position at some t has zero sensitivities to the configuration. If the market moves just a little, this state will be lost and we are exposed to some configuration risk. However, it is often too expensive to immediately rebalance the hedging portfolio: we would rather prefer to keep the sensitivity of the position in an acceptable region, and to obtain the cheapest hedging portfolio which allows this.

Portfolios under Constraints

Indeed, there are a couple of useful constraints when we try to find appropriate portfolio weights $\breve{\nu} = (\breve{\nu}^1, \ldots, \breve{\nu}^d)$. We denote by

$$\mathbb{O}(\breve{\nu})(\xi) := -\breve{H}(\xi) + \sum_{k=1}^{d} \breve{\nu}^k \, \breve{C}^k(\xi)$$

the value function of the respective overall portfolio.

(a) **Risk constraints**: For each ξ^k, we want to specify a "tolerance" $\tau^k \geq 0$ such that the exposure to this configuration value does not exceed τ^k:

$$|\partial_k \mathbb{O}(\breve{\nu})| \leq \tau^k \ . \tag{5.12}$$

(b) **Position constraints**: We want to avoid too large transactions sizes. Hence, we assume there are $L^\ell \leq 0 \leq U^\ell$ such that

$$L^\ell \leq \breve{\nu}^\ell \leq U^\ell \ . \tag{5.13}$$

(c) **Model error constraint:** Additionally, we want to ensure that the overall pricing error (mismatch between model prices and market prices for the liquid instruments) does not exceed a certain absolute bound ϵ,

$$\left| \sum_{\ell=1}^{d} \left(C^k - \breve{C}^k \right) \breve{\nu}^k \right| \leq \epsilon \ . \tag{5.14}$$

An extension to the sum of the absolute values or the supremum of all differences is also possible.

Finally, we assume that in order to buy or sell one of the liquid instruments $\mathbb{C} = (\mathbb{C}^1, \ldots, \mathbb{C}^d)$, we have to pay proportional transaction costs $c = (c^1, \ldots, c^d) \in (\mathbb{R}^+)^d$. Hence, the cost of building our hedging-portfolio is

$$\Pi(\breve{\nu}) := \sum_{\ell=1}^{d} c |\breve{\nu}^d| \ . \tag{5.15}$$

PROBLEM:
Find the weights ν which minimize (5.15) under the constraints (a) - (c).

This can be achieved using a linear programming algorithm. To this end, note that the standard form of an LP program we consider here is given as

$$\left.\begin{aligned} \text{minimize}_{w\in\mathbb{R}^q}\ w'c \\ l \le Aw \le u \end{aligned}\right\} \tag{5.16}$$

where $c \in \mathbb{R}^q$ is called the "cost vector", where $A \in \mathbb{R}^{p\times q}$ is a matrix and where $l, u \in \mathbb{R}^p$ are the constraints with $-\infty \le l^i \le u^i \le \infty$ for $i = 1, \dots, p$. Such a problem can be solved very efficiently, see for example Fang et al. [FP93].

We will now show that our problem can be translated into this form:

(a) **Risk constraints**:
To see that the risk-constraints (a) are linear in $\breve{\nu}$, we use a standard trick in linear programming: add the slack variables $s^1, \dots, s^K$ which will represent

$$s^k = \partial_k \mathbb{O}(\breve{\nu}) \ .$$

To enforce this equality, we add the $k = 1, \dots, K$ linear constraints

$$\partial_k \breve{H} = -s^k + \partial_k \breve{C}^1 \breve{\nu}^1 + \cdots + \partial_k \breve{C}^d \breve{\nu}^d \ .$$

To enforce (5.12), we then also add the constraint

$$-\tau^k \le s^k \le \tau^k \ .$$

(b) **Position Constraints**
These are already linear.

(c) **Model error constraint**
Inequality (5.13) is also implemented by using a slack variable, say u, which will represent the sum of the differences above. Indeed, the constraint

$$0 = -u + (C^1 - \breve{C}^1)\breve{\nu}^1 + \cdots + (C^d - \breve{C}^d)\nu^d$$

and the additional constraint

$$-\epsilon \le u \le \epsilon$$

implement together (5.13).

(d) **Optimization Target**
Since a standard LP algorithm tries to find an optimal vector x with respect to some cost-vector

$$x'c = x^1 c^1 + \cdots + x^d c^d \ ,$$

we need to introduce appropriate slack variables $x^\ell \ge 0$, which will represent $|\nu^\ell|$. To this end we impose the constraints

$$0 \le x^\ell - \nu^\ell \quad \text{and} \quad 0 \le x^\ell + \nu^\ell \ .$$

As a result, we will optimize over the expression

$$x'c$$

instead of $\nu'c$.

This gives us a linear programming problem of the form (5.16) for the vector

$$w' = (\nu^1, \ldots, \nu^K; \; s^1, \ldots, s^K; \; u; \; x^1, \ldots, x^d) ,$$

which will be optimized along

$$c' = (0, \ldots, 0; \; 0, \ldots, 0; \; 0; c^1, \ldots, c^d) .$$

The matrix A and the vectors l and u can be obtained from the above remarks.

REMARK 5.11 *The above program minimizes investment costs given a sensitivity constraint. This formulation has the advantage of a very clear meaning for all involved constraints and costs.*

However, a similar program can be developed which minimizes the sum of the sensitivities given a cost constraint. However, it is not clear to us how the different sensitivities can be scaled in a natural way such that a minimization of the sum of the sensitivities makes economic sense.

5.3 Dynamic Arbitrage

We now come to the issue of dynamic arbitrage in the meta-model of the institution following definition 5.7 of page 72. Recall that *dynamic arbitrage* was defined as a value process of an admissible payoff which is not a local martingale under any local martingale measure equivalent to the market measure $\mathbb{P}$ (cf. definition 5.9).

We will now again consider Heston's model as introduced in (5.1) on page 67:

$$\left.\begin{array}{rcl} d\bar{\zeta}_\tau &=& \bar{\kappa}(\bar{\theta} - \bar{\zeta}_\tau)\, dt + \bar{\nu}\sqrt{\bar{\zeta}_t}\, d\bar{W}^1_\tau \qquad \bar{\zeta}_0 = \zeta_0 \\ d\bar{X}_t &=& \sqrt{\bar{\zeta}_t}\, d(\bar{\rho}\bar{W}^1_\tau + \sqrt{1-\bar{\rho}^2}\bar{W}^2_t) \\ \bar{S}_\tau &=& S_0\, \mathcal{E}_\tau(\bar{X}) \; . \end{array}\right\} \tag{5.17}$$

Its configuration vector is $\xi = (\zeta_0; \kappa, \theta, \nu, \rho)$. We already know from example 3.3 on page 38 that the shape of the variance swap curve function

$$G(z; x) := z_2 + (z_1 - z_2)e^{-z_3 x} \; . \tag{5.18}$$

of this model (with $z_1 = \zeta_0$, $z_2 = \theta$ and $z_3 = \kappa$) implies that there is no consistent diffusion $Z = (Z^1, Z^2, Z^3) \in \Xi$ on *any* stochastic base such that the forward variance

$$G(Z_t; T - t)$$

is a local martingale.

Indeed, it also true that there is no continuous semi-martingale at all such that Z^3 is random. To this end, assume that $Z^i = M^i + A^i$ where M^i is a local martingale on the market space $\mathbb{W}$ and where A^i is of finite variation. Note that as before

$$\begin{aligned} dG(Z_t; T-t) &= -\partial_x G(Z_t; T-t)\, dt \\ &\quad + \partial_z G(Z_t; T-t)\; (dM_t + dA_t) \\ &\quad + \frac{1}{2}\partial^2_{zz} G(Z_t; T-t)\, d\langle M \rangle_t \; , \end{aligned}$$

which once more implies that

$$\partial_x G(Z_t; T-t)\,dt = \partial_z G(Z_t; T-t)\,dA_t + \frac{1}{2}\partial^2_{zz} G(Z_t; T-t)\,d\langle M\rangle_t$$

must hold $\mathbb{P}\times\lambda$-almost sure if $G(Z_t; T-t)$ is to be a local martingale. In case of (5.18), this means that

$$\begin{aligned}
-Z^3_t(Z^1_t - Z^2_t)e^{-Z^3_t x}\,dt &= -x(Z^1_t - Z^2_t)e^{-Z^3_t x} dA^3_t \\
&\quad + x^2(Z^1_t - Z^2_t)e^{-Z^3_t x} d\langle M^3\rangle_t \\
&\quad + (\cdots)e^{-Z^3_t x} dt
\end{aligned}$$

where the terms in $(\cdots)$ do not contain the parameter x. This implies first $\langle M^3\rangle_t \equiv 0$ and then $dA^3_t \equiv 0$, hence that Z^3 is constant. The same arguments also apply for a few other models introduced in chapter 3:

PROPOSITION 5.12 *A meta-model based on a model which has a polynomial-exponential variance curve functional (section 3.1) or an exponential mean-reverting curve functional (example 3.8) must have constant exponents to be free of dynamic arbitrage.*

Additionally, we will now prove:

PROPOSITION 5.13 (Dynamic Arbitrage in Heston's Model) *The meta-model based on Heston's model (5.1) is not free of dynamic arbitrage if the speed of mean-reversion, $\bar\kappa$, is not constant. The same is true if the product $\bar\rho\bar\nu$ is not constant.*

This requires the introduction of what we want to call an *entropy swap*. This concept is also of interest on its own; indeed, a close relative to the entropy swap, called "gamma swap" is now offered by several banks (see appendix A.1). The proposition is eventually proved on page 81.

5.3.1 Entropy Swaps

An *entropy swap* is a payoff very closely related to a variance swap. Under the assumption of Markovianity of (Z, S), an entropy curve functional similar to the variance curve functional will allow us to derive additional conditions on the possible volatility and correlation structure of a model.

The results will be put to use in the discussion of example 5.21 to show that the product of "volatility of variance" and correlation in the Heston model must remain constant.

DEFINITION 5.14 (Entropy Swap) *The payoff of an* entropy swap *with maturity $T<\infty$ is*

$$\int_0^T S_t\,d\langle \log S\rangle_t\ . \tag{5.19}$$

We denote its price by

$$U_t(T) := \mathbb{E}\left[\int_0^T S_t\,d\langle \log S\rangle_t \,\middle|\, \mathcal{F}_t\right]\ .$$

REMARK 5.15 *The payoff (5.19) is an approximation for the discretely sampled payoff*

$$\frac{d}{n}\sum_{i=1}^n \frac{S_{t_i}}{S_0}\left(\log\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2$$

where $0 = t_0 < \cdots < t_n = T$; compare equation (1.1) on page 10.

REMARK 5.16 (Gamma Swap) *In practise,* gamma swaps *are more popular than entropy swaps. If the forward curve of the underlying is constant, the two products coincide, but in general they are slightly different.*

Their relation is discussed in more detail in appendix A.1.2, where we also show how both entropy swaps and gamma swaps can be replicated using traded European options. An example term sheet for a gamma swap is provided on page 144.

Definition and basic Properties

Let $S_t := \mathcal{E}_t(X)$ be a true martingale with short variance ζ, i.e. $X_t := \int_0^t \sqrt{\zeta_s}\, dB_s$ for some $\mathbb{W}$-Brownian motion B.

Then, the stock price measure

$$\mathbb{P}^S[A] := \mathbb{E}\,[\, S_t 1_A \,] \qquad \text{for } A \in \mathcal{F}_t$$

cf. (2.15) is equivalent to $\mathbb{P}$. Consequently,

$$\frac{U_t(T)}{S_t} = \mathbb{E}^S \left[\int_0^T \zeta_s\, ds \,\middle|\, \mathcal{F}_t \right] \tag{5.20}$$

is a $\mathbb{P}^S$-martingale. Equation (5.20) rightly suggests that $U(T)$ can be interpreted as a variance swap under $\mathbb{P}^S$ (this is proved in appendix A.1.3).

Define the *forward entropy swap* curve $u = (u(T))_{T \geq 0}$ as

$$u_t(T) := \mathbb{E}\,[\, S_T \zeta_T \mid \mathcal{F}_t \,] \tag{5.21}$$

such that

$$U_t(T) = \int_0^T u_t(s)\, ds \ .$$

We also set

$$w_t(T) := \frac{u_t(T)}{S_t} \tag{5.22}$$

which is a $\mathbb{P}^S$-martingale for all finite T.

The name "entropy swap" is explained as follows: under the measure $\mathbb{P}^S$, the process $\tilde{B}_t = B_t - \int_0^t \sqrt{\zeta_s}\, ds$ is a Brownian motion. Hence

$$\log \frac{S_T}{S_t} = \int_t^T \sqrt{\zeta_s}\, d\tilde{B}_s + \frac{1}{2} \int_t^T \zeta_s\, ds \ , \tag{5.23}$$

so $\mathbb{E}^S[\int_0^T \zeta_s\, ds | \mathcal{F}_t] = 2\mathbb{E}^S[\log S_T/S_t | \mathcal{F}_t] + \int_0^t \zeta_s\, ds = 2/S_t\, \mathbb{E}[S_T \log S_T/S_t | \mathcal{F}_t] + \int_0^t \zeta_s\, ds$, which shows:

PROPOSITION 5.17

$$\frac{U_t(T)}{S_t} - \frac{U_t(t)}{S_t} = 2\,\mathbb{E}\left[\frac{S_T}{S_t} \log \frac{S_T}{S_t} \,\middle|\, \mathcal{F}_t \right] \ . \tag{5.24}$$

The entropy swap price $U_0(T)$ measures the entropy of $\mathbb{P}^S$ with respect to $\mathbb{P}$.

Compatible Entropy Curve Functionals

Assume that (G, Z, ρ) is a *strong* MVCMM as in definition 2.22. As in equation (2.24), we define the Brownian motion

$$B_t := \sum_{j=1}^{n} \int_0^t \rho^j(Z_s, S_s)\, dW_s^j \ ,$$

the short variance $\zeta_t := G(Z_t; 0)$ and the stochastic logarithm of S,

$$X_t := \int_0^t \sqrt{\zeta_s}\, dB_s \ .$$

Note that the process (Z, S) remains Markov under $\mathbb{P}^S$, with dynamics of Z given by

$$dZ_t = \mu^\rho(Z_t, S_t)\, dt + \sum_{j=1}^{n} \sigma^j(Z_t)\, d\tilde{W}_t^j \tag{5.25}$$

and with adjusted drift

$$\mu_i^\rho(z, s) := \mu_i(z) + \sqrt{G(z; 0)} \sum_{j=1}^{n} \rho^j(z, s)\sigma_i^j(z) \ . \tag{5.26}$$

According to proposition 2.10, equation (5.25) has a unique, non-explosive solution.

We have seen that w defined in (5.22) is a $\mathbb{P}^S$ martingale. Due to Markovianity of (Z, S), we here have

$$w_t(T) = H(Z_t, S_t; T - t) \tag{5.27}$$

for an *entropy curve functional*

$$H(z, s; x) := \mathbb{E}^S\left[\, G(Z_x; 0) \mid Z_0 = z, S_0 = s \,\right] \ . \tag{5.28}$$

In general, we may wish to *start* with G and H to find a pair (Z, ρ) such that all processes are well-defined. We precise this idea as follows:

DEFINITION 5.18 (Entropy Curve Functional) *An* entropy curve functional H *on an open set $\mathcal{Z} \subset \mathbb{R}^m$ is a positive $C^{2,2,2}$ function $H : \mathcal{Z} \times \mathbb{R}^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that $\int_0^T H(z, s; x)\, dx < \infty$ for all $T < \infty$.*

DEFINITION 5.19 (Compatible Entropy Swap Functionals) *An entropy curve functional H is called* compatible *with a consistent pair (G, Z) if there exists a* correlation function ρ *such that (G, Z, ρ) is a strong MVCMM and such that equation (5.28) is satisfied $\mathbb{P} \times \lambda|_{\mathbb{R}_0^+}$-almost surely.*

Note that then in particular $H(z, s; 0) = G(z; 0)$.

Because $w_t(T) = H(Z_t, S_t; T - t)$ is a martingale under $\mathbb{P}^S$ for all finite T, we have similar to theorem 2.24:

THEOREM 5.20 (HJM-condition for Entropy Swaps) *If H is compatible with (G, Z), then*

$$\left.\begin{aligned} \partial_x H(z, s; x) = \mu^\rho(z, s)\, \partial_z H(z, s; x) + \tfrac{1}{2}\sigma^2(z)\, \partial_{zz} H(z, s; x) \\ H(z, s; 0) = G(z; 0) \end{aligned}\right\} \ . \tag{5.29}$$

The adjusted drift μ^ρ is given in (5.26).

While we will use the previous theorem to constrain the possible dynamics of G, entropy swaps could also be used to infer additional information from the market. Indeed, the computations in section A.1.2 imply that we can infer the product of correlation and "volatility of variance" for a Heston model from the market.

Heston's Entropy Swap Functional

Under $\mathbb{P}^S$, the short variance in Heston's model (5.17) follows the SDE

$$\begin{aligned} d\bar{\zeta}_t &= \left(\bar{\kappa}\bar{\theta} - (\bar{\kappa} - \bar{\nu}\bar{\rho})\,\bar{\zeta}_t\right)\, dt + \bar{\nu}\sqrt{\bar{\zeta}_t}\, dW_t^{1S} \\ &= (\bar{\kappa} - \bar{\nu}\bar{\rho})\left(\frac{\kappa}{\bar{\kappa} - \bar{\nu}\bar{\rho}}\bar{\theta} - \bar{\zeta}_t\right)\, dt + \bar{\nu}\sqrt{\bar{\zeta}_t}\, dW_t^{1S} \\ &= \bar{c}\left(\frac{\bar{\kappa}}{\bar{c}}\bar{\theta} - \bar{\zeta}_t\right)\, dt + \bar{\nu}\sqrt{\bar{\zeta}_t}\, dW_t^{1S} \end{aligned}$$

where W^S is a $\mathbb{P}^S$-Brownian motion and with $\bar{c} := \bar{\kappa} - \bar{\nu}\bar{\rho}$. This is just a square-root diffusion with new mean-reversion speed and a new mean-reversion level. Hence, the price of an entropy swap in Heston's model is

$$\mathbb{E}^S\left[\,\zeta_x\,\right] = \frac{\bar{\kappa}}{\bar{c}}\bar{\theta} + \left(\bar{\zeta}_0 - \frac{\bar{\kappa}}{\bar{c}}\bar{\theta}\right) e^{-\bar{c}x}\ .$$

Example 5.21 (Heston's Entropy Swap Functional) *Let G be as in example 3.3 on page 38 and let*

$$H(z;x) = \frac{\kappa}{z_3}z_2 + \left(z_1 - \frac{\kappa}{z_3}z_2\right) e^{-z_3 x}\ . \tag{5.30}$$

Then, z_3 is a constant and we must have $\mu_1(z) = \kappa(z_2 - z_1)$, $\sigma_1(z)\rho(z) = (z_3 - \kappa)\sqrt{z_1}$, $\mu_2(z) = 0$ and $\sigma_2(z)\rho(z) = 0$.

Proof – We have seen in example 3.3 that $\mu_1(z) = \kappa(z_2 - z_1)$ and $\mu_2(z) = 0$. Since also $G(z;0) = H(z;0) = z_1$, theorem 5.20 implies that we have to match

$$\partial_x H(z;x) = \mu^\rho(z)\,\partial_z H(z;x) + \frac{1}{2}\sigma^2(z)\,\partial_{zz}H(z;x) \tag{5.31}$$

where

$$\mu_i^\rho(z) := \mu(z) + \sqrt{z_1}\sum_{j=1}^n \sigma_i^j(z)\rho^j(z)$$

for $i = 1, \ldots, 3$.

However, (5.31) has the same structure as an exponential-polynomial functional G. Therefore we can conclude immediately that Z^3 is a constant.

Consequently, (5.30) degenerates to example 3.3 i.e.

$$\mu_1^\rho(z) = z_3\left(\frac{\kappa}{z_3}z_2 - z_1\right) \quad \text{and} \quad \mu_2^\rho(z) = 0\ .$$

Combining the results yields the assertion. □

Proof of proposition 5.13– Since our previous results have already shown that $\bar{\kappa}$ must be constant in the meta-model, it follows that the product $\bar{\rho}\bar{\nu}$ must be constant if we want to avoid dynamic arbitrage. □

Part III

Practical Implementation

Chapter 6

A variance curve model

In this third part of the thesis we will discuss a concrete strong variance curve market model which is based on example 3.5. The model described here has been developed to price options on variance in particular: it tries to model the movement of the term structure of forward variance such that complex exotic options on variance such as options on forward variance swaps can be priced and hedged reliably.

If we propose a new model, we have to ensure first that it is properly defined and second that it can be put to use efficiently in practise. The first point means that we will check that the model actually defines a true martingale stock price process. Practicability is essentially equivalent to the ability to price and, often more challenging, calibrate the model to observed market data.

We begin with the formal definition of the model and the theoretical discussion. In section 6.2, we will then develop an efficient Monte-Carlo scheme which can be used to price exotic options. It is also used in the calibration which is discussed in chapter 7.

6.1 The Model

For $x \in \mathbb{R}$, $y \in [\frac{1}{2}, 1]$ and $\epsilon > 0$ define

$$x^{y,\epsilon} := (x^+ + \epsilon)^y - \epsilon^y \ . \tag{6.1}$$

This function is globally Lipschitz, vanishes in ϵ and is smooth on $[0, \infty)$ with first derivative $\partial_x x^{y,\epsilon} = y(x^+ + \epsilon)^{y-1}$. In particular, $\partial_x x^{y,\epsilon}|_{x=0} = \epsilon^{y-1}$ is finite. Figure 6.1 shows a few example curves for this function.

The variance curve model we propose is specified as follows: It is has the initial states

$$z = (\zeta_0, \theta_0, m_0)$$

and the parameter vector

$$\chi := (\kappa, c; \mu, \nu, \eta; \alpha, \beta, \gamma; \bar{m}; \rho_v, \rho_m, \rho_\theta, \rho_{v,\theta}) \ . \tag{6.2}$$

We call $\mu, \nu, \eta \in \mathbb{R}_{\geq 0}$ "ShortVolOfVol", "LongVolOfVol" and "InfVolOfVol". The initial levels $\zeta_0, \theta_0, m_0 \in \mathbb{R}_{>0}$ are referred to as "ShortVar", "LongVar" and "InfVar", while $\bar{m} \in [0, m_0)$ is called "InfVarFloor" and $\alpha, \beta, \gamma \in [\frac{1}{2}, 1]$ are named "ShortTwist", "LongTwist" and "InfTwist". The reversion speed $\kappa \in \mathbb{R}_{>0}$ is called "ShortRevSpeed" and $c \in \mathbb{R}_{>0}$ is called "LongRevSpeed". We also fix a floor $\epsilon > 0$, which we regard as constant, hence it is not a parameter.

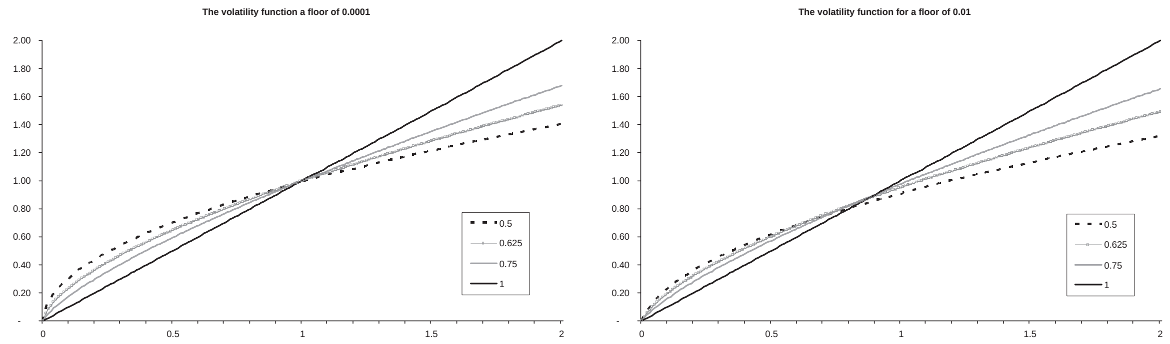


Figure 6.1: The function $x \mapsto (x^+ + \epsilon)^y - \epsilon^y$ for a floor ϵ of 0.0001 and 0.01.

The process $Z = (\zeta, \theta, m)$ is then the unique strong solution to the SDE

$$\left.\begin{array}{rcl} d\zeta_t &=& \kappa(\theta_t - \zeta_t)\,dt + \nu\zeta_t^{\alpha,\epsilon}\,d\hat{W}_t^{\zeta} \\ d\theta_t &=& c(m_t - \theta_t)\,dt + \mu\theta_t^{\beta,\epsilon}\,d\hat{W}_t^{\theta} \\ dm_t &=& \eta m_t^{\gamma,\bar{m}}\,d\hat{W}_t^{m} \end{array}\right\} \tag{6.3}$$

starting in (ζ_0, θ_0, m_0) where $m_0 > \bar{m} \geq 0$. We impose the following correlation structure in terms of a four-dimensional driving Brownian motion W:

$$\left.\begin{array}{rcl} B_t &=& W_t^1 \\ \hat{W}_t^{\zeta} &=& \rho_\zeta B_t + \hat{\rho}_\zeta W_t^2 \\ \hat{W}_t^{\theta} &=& \rho_\theta B_t + \hat{\rho}_\theta \left(r_{\zeta,\theta} W_t^2 + \hat{r}_{\zeta,\theta} W_t^3 \right) \\ \hat{W}_t^{m} &=& \rho_m B_t + \hat{\rho}_m W_t^4 \ , \end{array}\right\} \tag{6.4}$$

where we have used the convention $\hat{\rho} := \sqrt{1-\rho^2}$. We call $\rho_\zeta, \rho_\theta, \rho_m \in (-1, 0]$ "ShortCorrelation", "LongCorrelation" and "InfCorrelation" and $r_{\zeta,\theta} \in (-1, +1)$ "ShortLongCorrelation". The associated stock price is as usual given as

$$S_t := \mathcal{E}_t(X) \ , \quad X_t = \int_0^t \sqrt{\zeta_s}\,dB_s \ . \tag{6.5}$$

The variance curve functional G of this model is exactly (3.7) in example 3.5 on page 39 with $z = (\zeta_0, \theta_0, m_0)$:

$$G(z;x) := m_0 + (\zeta_0 - m_0)e^{-\kappa x} + (\theta_0 - m_0) \begin{cases} \frac{\kappa}{\kappa - c}\left(e^{-cx} - e^{-\kappa x}\right) & (\kappa \neq c) \\ \kappa x\, e^{-\kappa x} & (\kappa = c) \end{cases} \tag{6.6}$$

The model has the intuitive description of a curve which is described by a short-term factor ζ_t, a medium range factor θ_t and an infinite horizon factor m_t. Its variance swap price function $\mathbb{G}(z,x) := \int_0^x G(z;y)\,dy$ is

$$\mathbb{G}(z;x) = m_0 x + (\zeta_0 - m_0)\frac{1 - e^{-\kappa x}}{\kappa} + (\theta_0 - m_0) \begin{cases} \frac{\kappa}{\kappa - c}\left(\frac{1-e^{-cx}}{c} - \frac{1-e^{-\kappa x}}{\kappa}\right) & (\kappa \neq c) \\ \frac{1-(1+\kappa x)e^{-\kappa x}}{\kappa} & (\kappa = c) \end{cases} \tag{6.7}$$

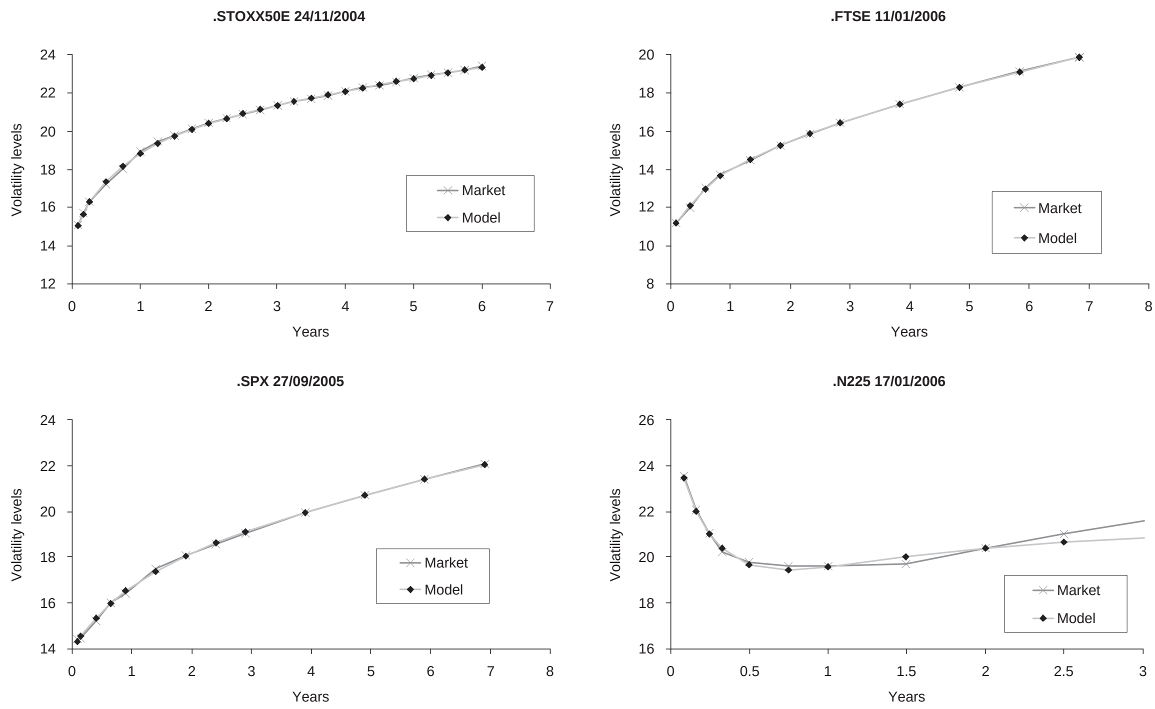


Figure 6.2: Fitting the variance curve (6.7) to market data. It generally fits well, even if the market experiences massive disruptions, as we can see in the example of the .N225 index (from Monday, January 16th 2006, to Wednesday the .N225 dropped by nearly 7% which in return drove up the prices of short term variance swaps). All graphs show "variance swap volatilites", cf. (1.2).

and has a suitable range of possible shapes: figure 3.1 shows a few calibrated curves (see also graphs on page 40). Moreover, figure 6.3 illustrates the impact of changing $(\zeta_0, \theta_0, m_0; \kappa, c)$ for the example of the FTSE curve. It shows how each of the states ζ, θ and m is responsible for a different part of the deformation of the curve.

The model is designed to capture the term-structure movements of the variance swap market prices, and thereby to allow pricing and hedging particularly of options on realized variance. For these products, the model describes with the "forward" of the underlying (realized variance) the natural hedging instrument.

Of course, the model can also be used to hedge other volatility-dependent products such as forward started options, but the lack of a "stochastic skew" parameter makes it more suited for the aforementioned products. Note, however, that such a "stochastic skew" could be introduced by making the correlation structure a function of the parameters[1] We will not discuss this approach here.

REMARK 6.1 *The process m will hit zero and will be absorbed there if either $\gamma > 0$ or $\bar{m} > 0$. Hence, we will typically assume that $\gamma = 1$, $\bar{m} = 0$.*

Correlation

Before we discuss existence and uniqueness, we want to write the correlation structure (6.4) in a more convenient way. Indeed, definition (6.4) above has been chosen because the involved

[1] The state of the respective new coordinate of Z could be inferred from market data using entropy swaps.

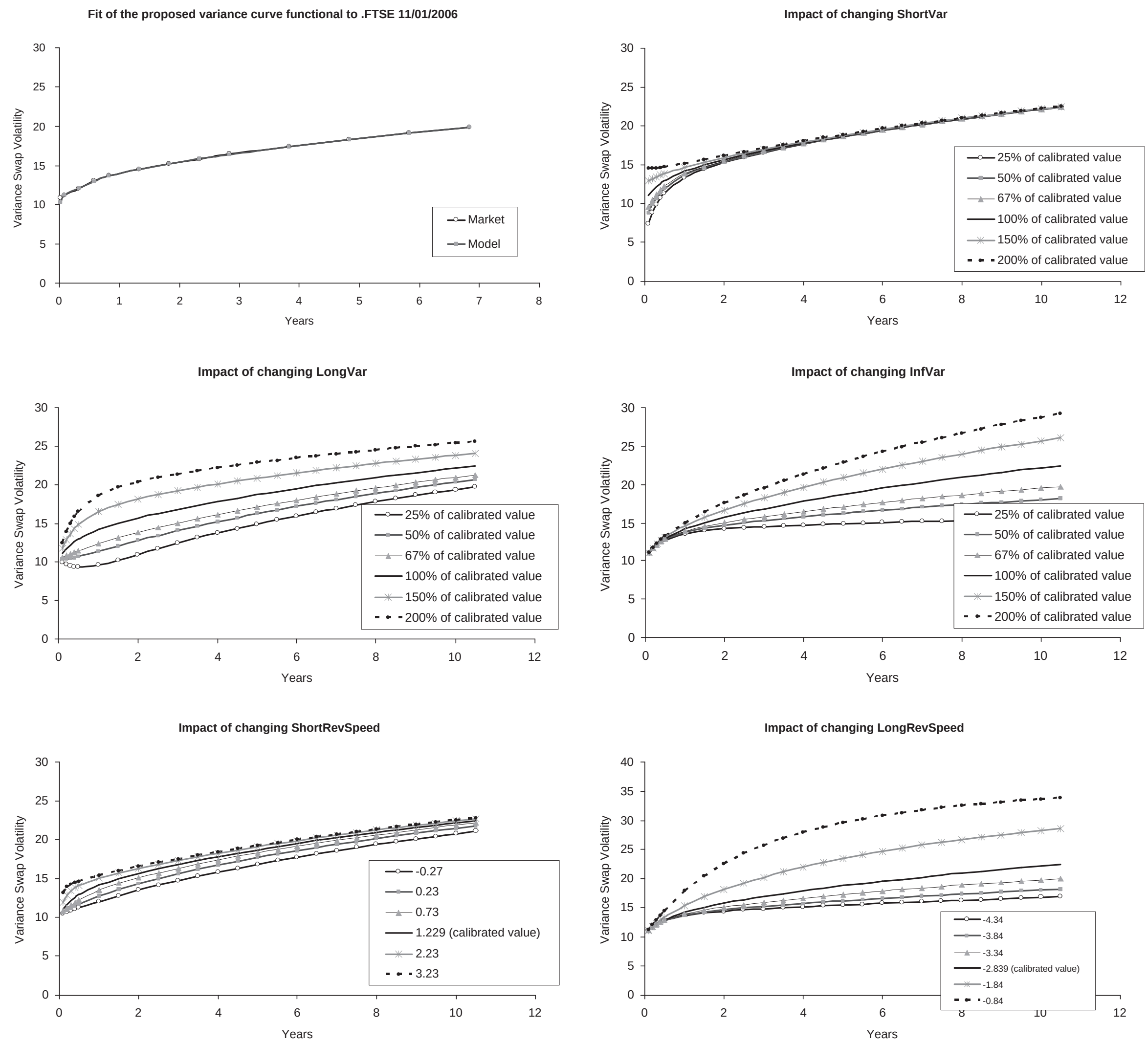


Figure 6.3: The impact of changes of the parameters to the curve calibrated to FTSE market data on January 11th, 2006; the top left graph recalls the fit to the market. The calibrated values for the FTSE are given in table 7.1 on page 121.

quantities have an intuitive meaning for the user and can be explained more easily. However, for ease of notation we shall henceforth consider the following mapping $W \mapsto (\hat{W}^\zeta, \hat{W}^\theta, \hat{W}^m, B)$, which is equivalent to (6.4) in distribution.

$$
\begin{aligned}
\hat{W}^\zeta_t &:= W^1_t \\
\hat{W}^\theta_t &:= \rho_{\zeta,\theta} W^1_t + \hat{\rho}_{\zeta,\theta} W^2_t \\
\hat{W}^m_t &:= \rho_{\zeta,m} W^1_t + \varrho_{\theta,m} W^2_t + \sqrt{1-\rho^2_{\zeta,m}-\varrho^2_{\theta,m}}\, W^3_t \\
B_t &:= \rho_\zeta W^1_t + \varrho_\theta W^2_t + \varrho_m W^3_t + \sqrt{1-\rho^2_\zeta-\varrho^2_\theta-\varrho^2_m}\, W^4_t \ ,
\end{aligned}
\tag{6.8}
$$

where we used

$$\begin{aligned}
\rho_{\zeta,\theta} &:= \langle \hat{W}^\zeta, \hat{W}^\theta \rangle_1 = \rho_\zeta \rho_\theta + r_{\zeta,\theta}\hat{\rho}_\zeta \hat{\rho}_\theta \\
\rho_{\zeta,m} &:= \langle \hat{W}^\zeta, \hat{W}^m \rangle_1 = \rho_\zeta \rho_m \\
\rho_{\theta,m} &:= \langle \hat{W}^\theta, \hat{W}^m \rangle_1 = \rho_\theta \rho_m \\
\varrho_{\theta,m} &:= (\rho_{\theta,m} - \rho_{\zeta,\theta}\rho_{\zeta,m})/\hat{\rho}_{\zeta,\theta} \\
\varrho_\theta &:= (\rho_\theta - \rho_\zeta \rho_{\zeta,\theta})/\hat{\rho}_{\zeta,\theta} \\
\varrho_m &:= (\rho_m - \rho_\zeta \rho_{\zeta,m} - \varrho_\theta \varrho_{\theta,m})/\sqrt{1 - \rho_{\zeta,m}^2 - \varrho_{\theta,m}^2} \ .
\end{aligned} \tag{6.9}$$

By making use of

$$\varrho_B := \sqrt{\rho_\zeta^2 + \varrho_\theta^2 + \varrho_m^2}$$

we can write the Brownian motion B as

$$B_t = \varrho_B B_t^1 + \hat{\varrho}_B B_t^2$$

in terms of the independent Brownian motions

$$B_t^1 := \frac{1}{\varrho_B}\left(\rho_\zeta W_t^1 + \varrho_\theta W_t^2 + \varrho_m W_t^3\right) \quad \text{and} \quad B_t^2 := W_t^4 \ .$$

6.1.1 Existence, Uniqueness and the Martingale Property

THEOREM 6.2 *The above model is a strong Markov Variance Curve Market Model.*

Before we prove this this statement, we will need to introduce the notion of a *process Lipschitz* operator, which is used in the comparison theorem 54 from Protter [P04], pg. 324, which will we cite below. We denote by $\mathbb{D}^n$ the space of n-dimensional adapted right continuous processes with left limits.

DEFINITION 6.3 (Process Lipschitz) *We call an operator* $F : \mathbb{D}^m \to \mathbb{D}^1$ process Lipschitz, *if for all* $X, Y \in \mathbb{D}^m$ *the following holds:*

(a) For all stopping times, $X_\tau = Y_\tau$ *implies* $F_\tau(X) = F_\tau(Y)$.

(b) There exists an adapted, left continuous process with right limits K *such that*

$$\|F(X)_t - F(X)_t\| \le K_t \|X_t - Y_t\| \ .$$

We need a less general formulation:

DEFINITION 6.4 (Parameter Lipschitz) *We call a continuous function* $\alpha : \mathbb{R}^\ell \times \mathbb{R}^m :\to \mathbb{R}$ parameter Lipschitz *if there exists some continuous function* $x \mapsto k(x)$ *such that*

$$\|\alpha(x; y_1) - \alpha(x; y_2)\| \le k(x) \|y_1 - y_2\| \ . \tag{6.10}$$

We call the function α monotonic parameter Lipschitz *if, additionally,* $\ell = 1$ *and if* $x \mapsto \alpha(x; y)$ *is strictly increasing.*

LEMMA 6.5 *Let α be parameter Lipschitz and assume that γ be a $\mathbb{R}^\ell$-valued diffusion. Then,*

$$F(\cdot) \equiv \alpha(\gamma; \cdot)$$

is process Lipschitz.

Proof – Set indeed $F(X)_t := \alpha(\gamma_t; X_t)$. Item (a) of definition 6.3 is satisfied by construction. Item (b) in turn is satisfied if we set $K_t := k(X_t)$ with k from (6.10). Note that $(K_t)_t$ is continuous. □

THEOREM 6.6 (Comparison theorem for random coefficients) *Let F^1, F^2 and $H^1, \ldots, H^d$ be process Lipschitz functionals such that $F_t^1(x) > F_t^2(x)$ for all x. Let X and Y be the solutions of*

$$\begin{aligned} dX_t &= F_t^1(X)\,dt + \sum_{j=1}^d H^j(X)_t\,dW_t^j \\ dY_t &= F_t^2(Y)\,dt + \sum_{j=1}^d H^j(Y)_t\,dW_t^j\ . \end{aligned}$$

If $X_0 \geq Y_0$, then $\mathbb{P}[\exists t > 0 : X_t \not> Y_t] = 0$, in other words X strictly dominates Y.

For a proof, see Protter [P04], pg. 324. We need a simplification which follows directly from lemma 6.5. We state it for the purpose of reference below.

COROLLARY 6.7 *Let α be monotonic parameter Lipschitz and let $b^1, \ldots, b^d$ be Lipschitz. Assume that γ^1 and γ^2 are one-dimensional diffusions and let X and Y solve*

$$dX_t = \alpha(\gamma_t^1; X_t)\,dt + \sum_{j=1}^d b^j(X_t)_t\,dW_t^j \tag{6.11}$$

$$dY_t = \alpha(\gamma_t^2; Y_t)\,dt + \sum_{j=1}^d b^j(Y_t)\,dW_t^j\ . \tag{6.12}$$

We then have

- *If $X_0 \geq Y_0$ and $\gamma^1 > \gamma^2$ almost surely, then $\mathbb{P}[\exists t \in [0, \tau] : X_t \not> Y_t] = 0$, in other words X strictly dominates Y.*
- *If $X_0 \geq Y_0$ and $\gamma^1 \geq \gamma^2$ almost surely, then $\mathbb{P}[\exists t \in [0, \tau] : X_t \not\geq Y_t] = 0$, in other words X dominates Y.*

With these preparations, we are now ready to prove theorem 6.2:

Proof of theorem 6.2– The proof is split up into in several steps.

Existence and Uniqueness

Using (6.8), we can write the SDE for $Z = (\zeta, \theta, m)$ as

$$dZ_t = \tilde{\mu}(Z_t)\,dt + \sigma(Z_t)\,dW_t$$

with

$$\tilde{\mu}(z) := \begin{pmatrix} \kappa(\theta - \zeta) \\ c(m - \theta) \\ 0 \end{pmatrix} \tag{6.13}$$

and

$$\sigma(z) = \begin{pmatrix} \nu\zeta^{\alpha,\epsilon} & 0 & 0 \\ \mu_1\theta^{\beta,\epsilon} & \mu_2\theta^{\beta,\epsilon} & 0 \\ \eta_1 m^{\gamma,\bar{m}} & \eta_2 m^{\gamma,\bar{m}} & \eta_3 m^{\gamma,\bar{m}} \end{pmatrix} \tag{6.14}$$

where we defined

$$\begin{aligned} \mu_1 &:= \mu\rho_{\zeta,\theta} \ , \\ \mu_2 &:= \mu\sqrt{1-\rho_{\zeta,\theta}^2} \ , \\ \eta_1 &:= \eta\rho_{\zeta,m} \ , \\ \eta_2 &:= \eta\varrho_{\theta,m} \ , \\ \eta_3 &:= \eta\sqrt{1-\rho_{\zeta,m}^2-\varrho_{\theta,m}^2} \ . \end{aligned}$$

Since $\alpha, \beta, \gamma \in [\frac{1}{2}, 1]$ and $\epsilon, \bar{m} > 0$ (or $\gamma = 1$ and $\bar{m} = 0$), it is clear that both $\tilde{\mu}$ and σ are globally Lipschitz, such that a unique strong non-explosive solution Z of (6.3) indeed exists (cf. [P04] theorem 7 pg. 264).

Non-Negativity of Z

Let m' be the solution to

$$dm'_t = \eta(m'_t)^{\gamma,\bar{m}}\, d\hat{W}_t^m$$

starting in $m'_0 = 0$. The solution is easily seen to be $m' \equiv 0$. Hence, $m_t \geq 0$ by monotony. Next, let θ' be the solution to

$$d\theta'_t = -c\theta'_t\, dt + \mu(\theta')^{\beta,\epsilon}\, d\hat{W}_t^\theta \tag{6.15}$$

again starting in $\theta'_0 = 0$. Let then $\alpha(x;y) := c(x-y)$, such that $\|\alpha(x,y_1) - \alpha(x,y_2)\| = \|cy_1 - cy_2\| = c\,\|y_1 - y_2\|$, i.e. α is monotonic parameter Lipschitz. We now obtain $\theta_t \geq 0$ by applying corollary 6.7 to the diffusions m and 0.

The same argument is used for $\bar{\alpha}(x;y) := \kappa(x-y)$ and the diffusions θ and 0 to prove that we also have $\zeta \geq 0$.

Remark 6.8 *If $\bar{m} = 0$ and $\gamma = 1$, then we have in fact $\zeta > 0$ and $\theta > 0$.*

Martingale-Property of v

We have to show that v with $v_t(T) := G(Z_t; T-t)$ is a martingale. We will actually show that it is a square-integrable martingale. Let

$$U(t) := \begin{cases} \frac{\kappa}{\kappa - c}\left(e^{-ct} - e^{-\kappa t}\right) & (\kappa \neq c) \\ \kappa t\, e^{-\kappa t} & (\kappa = c) \end{cases} \ . \tag{6.16}$$

Since (G, Z) are consistent, $v(T)$ is a supermartingale which satisfies

$$\begin{aligned} dv_t(T) = dG(Z_t; T-t) = \ & e^{-\kappa(T-t)}\nu\zeta_t^{\alpha,\epsilon}\, d\hat{W}_t^\zeta \\ & + U(T-t)\mu\theta_t^{\beta,\epsilon}\, d\hat{W}_t^\theta \\ & + \left(1 - e^{-\kappa(T-t)} - U(T-t)\right)\eta m_t^{\gamma,\bar{m}}\, d\hat{W}_t^m \ . \end{aligned}$$

According to corollary 1.25 in Revuz/Yor [RY99] it is sufficient to show that the quadratic variation of $v(T)$ has finite expectation for all finite $t \in [0,T]$. It is therefore sufficient to show that $Z = (\zeta, \theta, m)$ is square-integrable.

To this end, recall that if some process Y solves an SDE

$$dY_t = b(Y_t)\, dt + \sum_{j=1}^{d} \sigma^j(Y_t)\, dW_t^j$$

with Lipschitz b and $\sigma^1, \dots, \sigma^d$ for which there exists a constant K such that $\|b(x)\|_2^2 + \sum_{j=1}^d \|\sigma^j(x)\|_2^2 \le K(1+\|x\|_2^2)$, then $Y_T \in L^2(\mathbb{P})$ for all $T < \infty$, see Karatzas/Shreve [KS91] theorem 2.9, page 289.

This is the case for our model, hence the vector Z is in $L^2(\mathbb{P})$.

Martingale Property of S

We will prove that ζ does not explode under the measure $\mathbb{P}^S$ associated with the stock which implies that S is a true martingale, see proposition 2.10 on page 24. We adopt a method which is based on the proof of theorem 10.2.1 in Stroock/Varadhan [SV79] pg. 254.

(a) Under $\mathbb{P}^S$, the process $Z = (\zeta, \theta, m)$ has on $t < \tau$ the dynamics

$$dZ_t = \tilde{\mu}^\rho(Z_t)\, dt + \sigma(Z_t)\, dW_t^S$$

where W^S is a $\mathbb{P}^S$-Brownian motion and where

$$\tilde{\mu}^\rho(z) := \begin{pmatrix} \kappa(\theta - \zeta) + \rho_\zeta \nu \zeta^{\alpha,\epsilon} \sqrt{\zeta} \\ c(m-\theta) + \rho_\theta \mu \theta^{\beta,\epsilon} \sqrt{\zeta} \\ \rho_m \eta m^{\gamma,\bar{m}} \sqrt{\zeta} \end{pmatrix} . \tag{6.17}$$

Recall G from (6.6). The first derivatives are given as

$$\begin{aligned} \partial_\zeta G(z,t) &= e^{-\kappa t} , \\ \partial_\theta G(z,t) &= U(t) \quad \text{and} \\ \partial_m G(z,t) &= 1 - e^{-\kappa t} - U(t) \end{aligned} \tag{6.18}$$

with U as defined in (6.16). The second derivatives $\partial^2_{zz} G(z;t)$ of G vanish. By construction,

$$\partial_t G(z; T-t) + \tilde{\mu}(z) \partial_z G(z; T-t) + \frac{1}{2}\sigma^2(z)\, \partial^2_{zz} G(z; T-t) = 0 .$$

(b) The partial first derivatives (6.18) are non-negative: while this is clear for the first two derivatives, we show that $1 - e^{-\kappa t} - U(t) \ge 0$, too:

First assume $\kappa > c$.

$$\begin{aligned} (\kappa - c)\left(1 - e^{-\kappa t} - U(t)\right) &= \kappa - c - (\kappa - c)e^{-\kappa t} - \kappa\left(e^{-ct} - e^{-\kappa t}\right) \\ &= k - c + ce^{-\kappa t} - \kappa e^{-ct} \\ &= \kappa(1 - e^{-ct}) - c(1 - e^{-\kappa t}) \;\ge\; 0 . \end{aligned}$$

For $\kappa = c$, we have

$$1 - e^{-\kappa t} - U(t) \;=\; 1 - (1 + \kappa t)e^{-\kappa t} \ge 0 .$$

Finally, if $\kappa < c$, then also

$$(c - \kappa)\left(1 - e^{-\kappa t} - U(t)\right) \;=\; (c-\kappa)\left(1 - e^{-\kappa t}\right) + \kappa(e^{-ct} - e^{-\kappa t}) \;\ge\; 0 .$$

(c) Because the correlation factors ρ_ζ, ρ_θ and ρ_m are non-positive, we get

$$\begin{aligned}\tilde{\mu}^{\rho}(z)\partial_z G(z;T-t) &= \tilde{\mu}(z)\partial_z G(z;T-t)\\ &\quad + \rho_\zeta \nu \zeta^{\alpha,\epsilon}\sqrt{\zeta}\; e^{-\kappa(T-t)}\\ &\quad + \rho_\theta \mu \theta^{\beta,\epsilon}\sqrt{\zeta}\; U(T-t)\\ &\quad + \rho_m \eta m^{\gamma,\bar{m}}\sqrt{\zeta}\; (1-e^{-\kappa(T-t)}-U(T-t))\\ &\le \tilde{\mu}(z)\partial_z G(z;T-t)\ .\end{aligned}$$

Hence,

$$LG(z,T-t) := \partial_t G(z;T-t) + \tilde{\mu}^{\rho}(z)\partial_z G(z;T-t) + \frac{1}{2}\sigma'\sigma(z)\,\partial^2_{zz}G(z;T-t) \le 0$$

is non-positive.

(d) Let $\tau_n := \inf\{t : ||Z_t||_2 \ge n\}$ for $n \in \mathbb{N}$ and set $Z^n_t := Z_{t\wedge\tau_n}$. On $t \le T \wedge \tau_n$, we have

$$dG(Z^n_t;T-t) \;=\; LG(Z^n_t,T-t)\,dt + \partial_z G(Z^n_t;T-t)\,\sigma(Z^n_t)\,dW^S_t\ .$$

Since Z^n is bounded and the volatility term is locally bounded (i.e. bounded on compact intervals) in its z-argument, the expectation of the stochastic integral vanishes.

Hence,

$$\begin{aligned}G(Z_0;T) &= \mathbb{E}^S\left[G(Z^n_T;T-\{T\wedge\tau_n\}) - \int_0^{T\wedge\tau_n} LG(Z^n_t,T-t)\,dt\right]\\ &\ge \mathbb{E}^S\left[\,G(Z^n_T;T-\{T\wedge\tau_n\})\,\right]\\ &= \mathbb{E}^S\left[\,G(Z^n_T;T-\tau_n)1_{\tau_n\le T} + G(Z^n_T;0)1_{\tau_n>T}\,\right]\\ &\ge \mathbb{E}^S\left[\,G(Z^n_T;T-\tau_n)1_{\tau_n\le T/2}\,\right]\\ &\overset{(*)}{\ge} \mathbb{E}^S\left[\,IG(n)1_{\tau_n\le T/2}\,\right]\\ &= IG(n)\ \mathbb{P}^S[\ \tau_n \le T/2\ ]\end{aligned}$$

with

$$IG(n) := \inf\left\{\, G(z;T-t)\ \middle|\ (z,t) \text{ such that } t\in[0,T/2] \text{ and } \|z\|_2 = n \,\right\}\ .$$

Inequality $(*)$ follows because $\|Z^n_T\|_2 = n$ on $\{\tau_n \le T\}$.

In the final step, note from (6.6) that

$$\lim_{n\uparrow\infty}\ IG(n) = \infty\ .$$

Whence

$$\lim_{n\uparrow\infty}\ \mathbb{P}^S[\ \tau_n \le T/2\ ] = 0$$

must hold to satisfy $IG(n)\ \mathbb{P}^S[\ \tau_n \le T/2\ ] < \infty$ above.

Completeness

Proposition 4.5 on page 53 applies, hence the model weakly preserves smoothness. Moreover, the functional $\mathbb{G}$ is τ-invertible for any $0 < \tau < x_1 < x_2 < x_4$.

This concludes the proof. □

Remark 6.9 *We have proved that the non-positivity condition on the correlation parameters ρ_ζ, ρ_θ and ρ_m is sufficient to ensure that S is a true martingale.*

Necessity has not been shown and is not true in general, as reduction to Heston via $\nu = \mu = 0$ and $c = 0$ shows. However, in the multi-factor case, we are not aware of a conceptually straightforward framework such as the Feller test on explosion, which can be employed for the classic one-factor stochastic volatility case.

Lewis discusses this approach in his book [L00] and it is applied to a range of stochastic volatility models in Andersen/Piterbarg [AP04].

6.2 Pricing

The model (6.3) is a high-dimensional model for which we are not aware of closed form pricing formulas for options other than variance swaps. Given the relatively high dimensionality of the model, the only generic pricing method at our disposal is Monte-Carlo. Of course, the related PDE for the model (6.3) can be derived but such high-order systems cannot, to the best of our knowledge, be solved efficiently yet.

A relatively efficient Monte-Carlo scheme for our model can be implemented as we will show below. One reason is that we have imposed Lipschitz-continuity on the volatility functions of the parameters and so we can employ a Milstein-scheme with better convergence properties than the Euler scheme. In contrast, this can not be applied to Heston's model because of the particular shape of the volatility coefficient of the short variance in this model. Another advantage of our model is that the readily available variance swap prices can be used as efficient control variates.

All the following schemes will discretisise the process (S, ζ, θ, m) on a set $0 = t_0 < \cdots < t_M =: T$ of dates. We let $\Delta t_i := t_{i+1} - t_i$. We will use simple indices i instead of t_i to indicate approximated values. So, ζ_M is the approximated value of ζ_T. We also assume that $Y = (Y_i)_{i=1,\dots,M}$ is an iid-sequence of 4-dimensional normal variables with mean zero, covariance zero and variance Δt_i for $i = 1, \dots, M$.

Correlation Structure

The correlation structure (6.4) of the model will be implemented using (6.8) as explained above; recall the definitions in (6.9).

$$
\begin{aligned}
\Delta \hat{W}_i^\zeta &:= Y_i^1 \\
\Delta \hat{W}_i^\theta &:= \rho_{\zeta,\theta} Y_i^1 + \sqrt{1 - \rho_{\zeta,\theta}^2}\, Y_i^2 \\
\Delta \hat{W}_i^m &:= \rho_{\zeta,m} Y_i^1 + \varrho_{\theta,m} Y_i^2 + \sqrt{1 - \rho_{\zeta,m}^2 - \varrho_{\theta,m}^2}\, Y_i^3 \\
\Delta B_i &:= \varrho_B \Delta B_t^1 + \sqrt{1 - \varrho_B^2}\, Y_i^4
\end{aligned}
\tag{6.19}
$$

with

$$
\Delta B_i^1 := \frac{1}{\varrho_B} \left(\rho_\zeta Y_i^1 + \varrho_\theta Y_i^2 + \varrho_m Y_i^3 \right) .
$$

The sequence $(B_i^1)_{i=1,\dots,M}$ is a normal sequence (with mean zero and variance Δt_i), which is independent of the sequence $(Y_i^4)_{i=1,\dots,M}$.

The advantage of the above structure is that the sub-matrix of the first three rows and columns can be isolated, so the variance process can be simulated independently of the stock price with only three normals per interval.

The key observation is indeed that

$$X_t \stackrel{d}{=} X_t^1 + X_t^2 \tag{6.20}$$

with

$$X_t^1 := \varrho_B \int_0^t \sqrt{\zeta_s}\, dB_t^1 - \frac{1}{2}\varrho_B^2 \int_0^t \zeta_s\, ds \tag{6.21}$$

and

$$X_t^2 = \left(\sqrt{(1-\varrho_B^2) \int_0^t \zeta_s\, ds} \right) B_t^2 - \frac{1}{2}(1-\varrho_B^2) \int_0^t \zeta_s\, ds \tag{6.22}$$

where B^2 is independent from (ζ, θ, m, B^1).

Hence, if the payoff depends on S only on a few of the dates t_i (which is typically the case), we can considerably reduce the number of normals we need to generate by simulating only X^1 at all time-steps. The process X^2 can then be simulated with "big jumps". Moreover, we will see that we can avoid simulating X^2 alltogether when we price European options.

All this is discussed below.

6.2.1 Pricing General Payoffs using an unbiased Milstein Monte-Carlo Scheme

The *Euler-scheme* for (6.3) reads

$$\left.\begin{array}{rcl}
\zeta_{i+1} - \zeta_i &=& \kappa(\theta_i - \zeta_i)\,\Delta t_i + \nu \zeta_i^{\alpha,\epsilon}\,\Delta \hat{W}_i^\zeta \\
\theta_{i+1} - \theta_i &=& c(m_i - \theta_i)\,\Delta t_i + \mu \theta_i^{\beta,\epsilon}\,\Delta \hat{W}_i^\theta \\
m_{i+1} - m_i &=& \eta m_i^{\gamma,\bar{m}}\,\Delta \hat{W}_i^m \\
X_{i+1} - X_i &=& \sqrt{\zeta_i^+}\,\Delta B_t - \frac{1}{2}\zeta_i^+\,\Delta t_i
\end{array}\right\} \tag{6.23}$$

(here, we use $x^{y,\epsilon} := (x^+ + \epsilon)^y - \epsilon^y$ which is equal to the function defined above (6.1) as long as the processes are not negative).

The stock price itself is given as $S_i := e^{X_i}$ and need only be computed for dates t_i where the product depends on the stock price level (writing S as the exponential of X has the advantage that it can never get negative). The last equation for m can be altered for the common case $\gamma = 1$ and $\bar{m} = 0$, i.e. if m is a geometric Brownian motion: as for the stock we would then simulate $\log m_{i+1} - \log m_i = \eta\,\Delta \hat{W}_i^m - \frac{1}{2}\eta^2\,\Delta t_i$ which, by taking the exponential, ensures that m is strictly positive.

Moreover, if any of ν, μ or η is zero, we can omit the simulation of the respective random numbers: the time consumed by the generation of these numbers with standard methods as described in Press et al. [PTVF02] can easily dominate the speed of the scheme above, so it is important to reduce the number of normals generated if possible.

The Euler-scheme is a robust approach but also of rather poor convergence. It also has a bias for each of the variables X, ζ, θ, m. We therefore improve upon (6.23) by using an unbiased Milstein scheme.

We first review the classical Milstein scheme.

Milstein scheme

The Euler-scheme (6.23) is of strong order 0.5 according to Kloeden/Platen [KP99] chapter 10.2. It can be improved by using the *Milstein scheme* (chapter 10.3 in [KP99]): for a general SDE

$$U_t = U_0 + \int_0^t a(s;U_s)\,ds + \sum_{j=1}^{d}\int_0^t b^j(s;U_s)\,dW_s^j \tag{6.24}$$

with $U \in \mathbb{R}^m$ and correlated d-dimensional Brownian motion W, the quality of convergence is dominated by the error of the approximation of the stochastic integral. The idea is therefore to improve the accuracy of the approximation of the $b(s;U_s)dW_s$ term using

$$\begin{aligned}
b^j(t;U_t) &= b^j(0;U_0) + \sum_{i=1}^{m}\int_0^t \partial_{u_i} b^j(s;U_s)\,dU_s^i + (\cdots)\,dt \\
&\overset{(6.24)}{=} b^j(0;U_0) + \sum_{\ell=1}^{d}\left(\sum_{i=1}^{m}\int_0^t b_i^\ell(s;U_s)\partial_{u_i} b^j(s;U_s)\right)\,dW_s^\ell + (\cdots)\,dt \\
&=: b^j(0;U_0) + \sum_{\ell=1}^{d}\int_0^t \beta^{j,\ell}(s;U_s)\,dW_s^\ell + (\cdots)\,dt \\
&\approx b^j(U_0) + \sum_{\ell=1}^{d}\beta^{j,\ell}(0;U_0)\,W_t^\ell
\end{aligned}$$

where we have set $\beta^{j,\ell}(t,u) := \sum_{i=1}^m b_i^\ell(t;u)\partial_{u_i} b^j(t;u)$. We obtain

$$\int_0^T b^j(t;U_t)dW_t^j \approx b^j(0;U_0)W_T^j + \sum_{\ell=1}^{d}\beta^{j,\ell}(0;U_0)\int_0^T W_t^\ell\,dW_t^j\ . \tag{6.25}$$

The *Milstein scheme* for (6.24) is now given as

$$U_{i+1} - U_i = a(t_i;U_i)\,\Delta t_i + \sum_{j=1}^{d} b^j(t_i;U_i)\,\Delta W_t^j + \sum_{j,\ell=1}^{d}\beta^{j,\ell}(t_i;U_i)I_{\ell,j}(i)$$

where $I_{\ell,j}(i) := \int_{t_i}^{t_{i+1}} W_t^\ell\,dW_t^j$. These mixed stochastic integrals $I_{\ell,j}$ are generally not easy to simulate accurately except for the case $\ell = j$ where we simply have

$$\int_0^T W_t^\ell\,dW_t^\ell = \frac{1}{2}((W_T^\ell)^2 - T)$$

(cf. [KP99] chapter 10.3). In our case, the mixed terms $b_i^\ell(0;U_0)\partial_{u_i} b^j(0;U_0)$ for $\ell \neq j$ of the variance process (ζ,θ,m) are zero so we can implement the following scheme efficiently:

$$\left.\begin{aligned}
\zeta_{i+1} - \zeta_i &= \kappa(\theta_i - \zeta_i)\,\Delta t_i + \nu\zeta_i^{\alpha,\epsilon}\,\Delta\hat{W}_i^\zeta + \tfrac{1}{2}\nu^2\bar{\zeta}^{\alpha,\epsilon}\left((\Delta\hat{W}_i^\zeta)^2 - \Delta t_i\right) \\
\theta_{i+1} - \theta_i &= c(m_i - \theta_i)\,\Delta t_i + \mu\theta_i^{\beta,\epsilon}\,\Delta\hat{W}_i^\theta + \tfrac{1}{2}\mu^2\bar{\theta}_i^{\beta,\epsilon}\left((\Delta\hat{W}_i^\theta)^2 - \Delta t_i\right) \\
m_{i+1} - m_i &= \eta m_i^{\gamma,\bar{m}}\,\Delta\hat{W}_i^m + \tfrac{1}{2}\eta^2\bar{m}_i^{\gamma,\bar{m}}\left((\Delta\hat{W}_i^m)^2 - \Delta t_i\right)
\end{aligned}\right\} \tag{6.26}$$

where we used the notation

$$\bar{x}^{y,\epsilon} := y\frac{(x^+ + \epsilon)^y - \epsilon^y}{(x^+ + \epsilon)^{1-y}}\ .$$

This function is well-defined for all $x \in \mathbb{R}_{\geq 0}$ since $\epsilon > 0$ (the idea does not work for Heston's model since the above function is not defined in zero for $y = \frac{1}{2}$ and $\epsilon = 0$). We obtain a strong order 1 scheme for the variance process. Figure 6.4 shows the improved convergence for the pricing of a call.

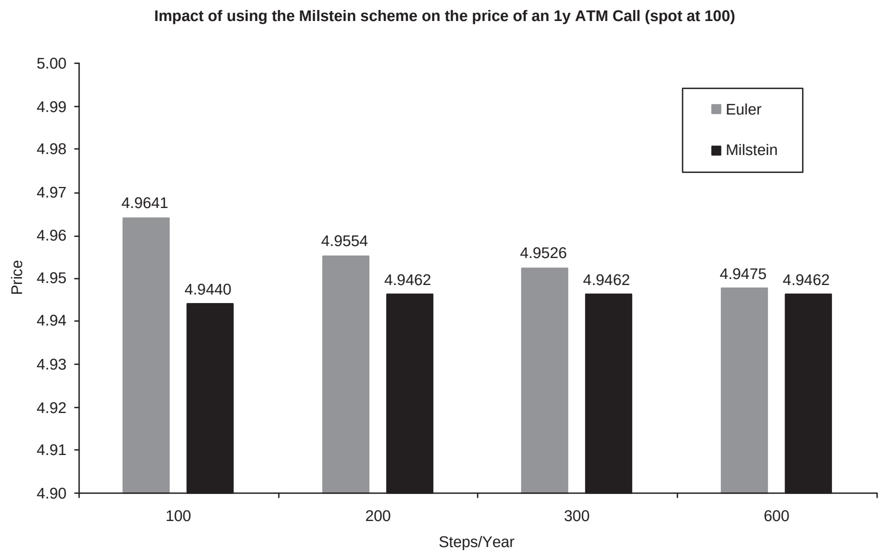


Figure 6.4: The price of an 1y ATM call with Euler and Milstein, respectively. We can clearly see the improved convergence with the Milstein scheme.

Bias reduction

The above discretization scheme naturally has a bias: the expectation of θ_M for example is

$$\mathbb{E}\left[\,\theta_M\,\right] = m_0 + (\theta_0 - m_0) \prod_{i=0}^{M-1} (1 - c\Delta t_i)$$

which differs from the theoretical result

$$\theta_T = m_0 + (\theta_0 - m_0)e^{-cT} \ .$$

A similar problem obviously appears for ζ. Note that this will happen regardless of the volatility structure of the processes: in particular, if the "VolOfVol" terms of all variables ζ, θ and m are zero we still have a discretization error even though model (6.3) just reduces to a log-normal stock price model.

This is a general drawback of discretization methods, but it is of particular importance here because we want to use the analytical variance swap prices as control variates: for this, we have to ensure that the Monte-Carlo scheme actually converges to those prices.

To this end, note that θ_t is given as

$$\theta_t = m_0 + (\theta_0 - m_0)e^{-ct} + \mu \int_0^t e^{-c(t-s)} \theta_s^{\beta,\epsilon} \, d\hat{W}_s^\theta \ . \tag{6.27}$$

If we now use the drift above instead of the discretization used in (6.26), then the estimator for θ will be unbiased.[2] We extend the idea to ζ, where

$$\zeta_t = G(\zeta_0, \theta_0, m_0; t) + \nu \int_0^t e^{-c(t-s)} \zeta_s^{\alpha,\epsilon}\, d\hat{W}_s^\zeta \ . \tag{6.28}$$

with G defined in (6.6).

As before, we will improve the convergence of the stochastic integral term by applying the Milstein scheme (6.25). The exponential term $e^{-c(t-s)}$ in the volatility terms drops out of the relevant expressions and we obtain the same volatility coefficients as in (6.26).

The scheme we shall employ is

$$\left.\begin{array}{rcl} \zeta_{i+1} & = & G(\zeta_i, \theta_i, m_i; \Delta t_i) + \nu \zeta_i^{\alpha,\epsilon}\, \Delta\hat{W}_i^\zeta + \frac{1}{2}\nu^2 \bar{\zeta}^{\alpha,\epsilon} \left((\Delta\hat{W}_i^\zeta)^2 - \Delta t_i \right) \\ \theta_{i+1} & = & m_i + (\theta_i - m_i) e^{-c\Delta t_i} + \mu \theta_i^{\beta,\epsilon}\, \Delta\hat{W}_i^\theta + \frac{1}{2}\mu^2 \bar{\theta}_i^{\beta,\epsilon} \left((\Delta\hat{W}_i^\theta)^2 - \Delta t_i \right) \\ m_{i+1} - m_i & = & \eta m_i^{\gamma,\bar{m}}\, \Delta\hat{W}_i^m + \frac{1}{2}\eta^2 \bar{m}_i^{\gamma,\bar{m}} \left((\Delta\hat{W}_i^m)^2 - \Delta t_i \right) \ . \end{array}\right\} \tag{6.29}$$

Note that $\mathbb{E}[\Delta\hat{W}_i^\cdot + ((\Delta\hat{W}_i^\cdot)^2 - \Delta t_i)] = 0$, which shows that this scheme is indeed unbiased.

In summary, the model (6.3) can reasonably efficiently be simulated using (6.29) for the variance. Figure 6.5 shows the result of removing the bias.

Simulating X

Let now $\mathcal{T} := \{\tau_1, \ldots, \tau_K\} \subset \{t_1, \ldots, t_M\}$ be the dates where the payoff explicitly depends on S. Also set $\tau_0 = 0$. We simulate for all $t_1 < \cdots < t_M$ the process Z using (6.20). Additionally, we also simulate

$$\mathcal{V}_t := \int_0^t \zeta_s\, d_s \quad \text{and} \quad X_t^1 := \varrho_B \int_0^t \sqrt{\zeta_s}\, dB_s^1 - \frac{1}{2}\varrho_B^2 \int_0^t \zeta_s\, ds \ .$$

See also (6.21) and the correlation structure defined in (6.19). Using an unbiased Euler-scheme, we approximate

$$\int_{t_i}^{t_{i+1}} \zeta_t\, dt \approx \mathbb{E}\left[\int_{t_i}^{t_{i+1}} \zeta_t\, dt \,\middle|\, Z_{t_i} \right] = \mathbb{G}(\zeta_i, \theta_i, m_i; \Delta t_i) \ ,$$

where $\mathbb{G}$ is the variance swap price function (6.7). We then set

$$\begin{array}{rcl} X_{i+1}^1 - X_i^1 & = & \sqrt{\varrho_B^2 \dfrac{\mathbb{G}(\zeta_i, \theta_i, m_i; \Delta t_i)}{\Delta t_i}}\, \Delta B_i^1 - \dfrac{1}{2}\varrho_B^2 \mathbb{G}(\zeta_i, \theta_i, m_i; \Delta t_i) \\ \mathcal{V}_{i+1} - \mathcal{V}_i & = & \mathbb{G}(\zeta_i, \theta_i, m_i; \Delta t_i) \end{array}$$

[2] In the affine case $\beta = 1/2$ and $\epsilon = 0$, we can exploit that the variance of θ is known to be:

$$\begin{array}{rcl} \mu^2 \displaystyle\int_0^t e^{-2c(t-s)} \mathbb{E}\left[\sqrt{\theta_s}^2 \right] ds & = & \mu^2 \displaystyle\int_0^t e^{-2c(t-s)} \left(m_0 + (\theta_0 - m_0) e^{-cs} \right) ds \\ & = & \mu^2 m_0 \dfrac{1 - e^{-2ct}}{2c} + \mu^2 (\theta_0 - m_0) \dfrac{e^{-ct}(1 - e^{-ct})}{c} \end{array}$$

In this case, using

$$\theta_t \approx m_0 + (\theta_0 - m_0) e^{-ct} + \mu \sqrt{m_0 \frac{1 - e^{-2ct}}{2c} + (\theta_0 - m_0) \frac{e^{-ct}(1 - e^{-ct})}{c}}\, \hat{W}_t^\theta$$

is also moment-matching the second moment. This is useful when we want to simulate a Heston-type model. However, for general β or $\epsilon > 0$ this is not applicable.

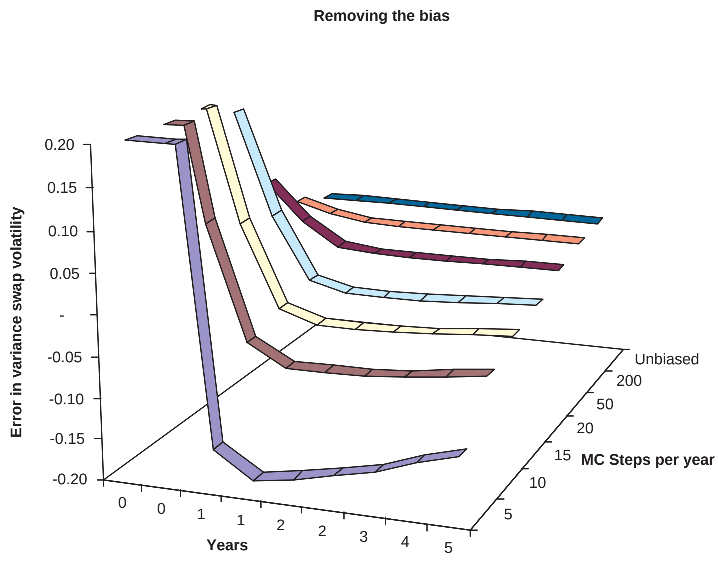


Figure 6.5: The error in variance swap volatilities $\sqrt{V_0(T)/T}$ when used with the biased scheme (6.26). In contrast, the terminal data set (which is labeled "unbiased") was computed using (6.29) with just one step per year.

Assume now that we need to know the value of the stock price for some $\tau_k = t_i > 0$. Let $\tau_{k-1} = t_j$ and compute

$$X_i^2 - X_j^2 = \sqrt{(1-\varrho_B^2)\Big(\mathcal{V}_i - \mathcal{V}_j\Big)}\, Y_i^4 - \frac{1}{2}(1-\varrho_B^2)\Big(\mathcal{V}_i - \mathcal{V}_j\Big)$$

such that we get

$$X_i = X_i^1 + X_i^2 \ .$$

The stock price is accordingly given by

$$S_i := \exp X_i \ .$$

The above construction ensures that S_i and the expected realized variance of S are unbiased.

Weak Approximations

In Kloeden/Platen [KP99], it is discussed that we actually do not really need to use standard normal increments Y in (6.19) for the simulation of Z. Instead, we can use a sequence of iid-variables which match the first and second moment of the Brownian path. In particular, we can use independent binary variables $(Y_i^1, \dots, Y_i^3)$ which are ± 1 with probability $p = 1/2$ for each coordinate (the stock price random variable should remain normal if we discretizise it only at the dates $\tau_1, \dots, \tau_K$). It is far quicker to simulate such binary variables; also see Bruti Liberati/Platen [BLP04].[3]

In such a simulation, the solutions converge *in distribution*, or *weakly*. Since the scheme (6.29) will have to be implemented on a reasonably fine time-grid anyway, such an approach is very useful if we want to price structured products; in practise, however, it is advisable to start

[3] Given that we will take big steps with Y^4, we recommend to use proper normal variables for that variable.

with normal increments for the first few steps to ensure that the resulting paths have sufficient variance.

6.2.2 Control Variates

From our experience, among the most powerful tools to improve the convergence of a Monte-Carlo scheme is the use of *control variates*. A good reference on the subject is Glasserman [Gl04].

The general idea is as follows: suppose we want to compute the expectation of some square-integrable random variable X with values in $\mathbb{R}$ on some probability space. We are not able to compute the true expectation $x := \mathbb{E}[\,X\,]$, but we can compute an unbiased approximation x_N by use of some Monte-Carlo scheme with N paths. Such an approximation will converge to a normal with mean x and variance $\mathrm{Var[X]}/N$ by the law of large numbers. Therefore, the efficiency of the estimation can be improved if variance can be reduced.

Assume that there is another square-integrable variable Y with values in $\mathbb{R}^d$, for which we know the vector $y := \mathbb{E}[\,Y\,]$ analytically (in the model (6.3) this will be the expected stock price and prices of variance swaps). On the other hand, we can also use our Monte-Carlo scheme to compute an approximated price vector y_N.

The general idea is now that if Y and X are "close", it should "reduce the variance" (understood as "uncertainty") if we instead approximate the value of

$$Z := X - \sum_{i=1}^{d} \varrho_i Y^i \ ,$$

where ϱ is some suitable vector.

Then, a new approximation for x can be computed using

$$\tilde{x}_N := z_N + \sum_{i=1}^{d} \varrho_i y^i$$

where, as we recall, y^i is the analytically known expectation of Y^i. This new approximation is obviously more efficient if the variance of Z is lower than the variance of X. We have

$$\mathrm{Var[X} - \sum_{\mathrm{i=1}}^{\mathrm{d}} \varrho_{\mathrm{i}}\mathrm{Y^i]} = \mathrm{Var[X]} - 2\sum_{i=1}^{d} \varrho_i \mathrm{Cov[X, Y^i]} + \sum_{i,j=1}^{d} \varrho_i \varrho_j \mathrm{Cov[Y^i, Y^j]} \ .$$

This is obviously minimized if ϱ is the vector

$$\varrho := \mathrm{Cov[Y^i, Y]}^{-1}\mathrm{Cov[X, Y]} \ .$$

In other words, instead of pricing X with the Monte-Carlo scheme, we only price the "orthogonal" part of X with respect to Y.[4]

In general, of course, we will not know ϱ analytically. However, it can be estimated using standard estimators from the same path which is used to compute x_N and y_N. For example,

$$\begin{aligned}
\overline{XY^i}_N \quad &:= \quad \frac{1}{N}\sum_{j=1}^{N} \hat{X}_{j,N}\hat{Y}^i_{j,N} \\
\mathrm{Cov[X, Y^i]}_N \quad &:= \quad \frac{1}{N-1}\left(\sum_{j=1}^{N} \hat{X}_{j,N}\hat{Y}^i_{j,N} - \overline{XY^i}_N\right)
\end{aligned}$$

[4]The quotes around "orthogonal" should alert that orthogonality and zero correlation are only the same if X and Y are jointly Gaussian.

where we have denoted by $\hat{X}_{j,N}$ the simulated value of X in the jth simulation for $j = 1, \dots, N$.

In (6.3), a natural control variate is the stock price itself with $\mathbb{E}[S_t] = 1$, but also the variance swap prices $V_0(T) := \mathbb{G}(Z_0; T)$, which are quick to compute. It is better to use the integrated prices $V_0(T)$ than the expected forward variances $v_0(T)$ since the price of most products depend far more on the former variable: the instantaneous variance is of little effect, while the integrated variance is a good measure of the width of the distribution of the returns of the stock price.

Indeed, another useful control variate is given if we use the deterministic variance $V_0(T)$ and integrate it over the Brownian motion B. This yields a Black&Scholes geometric Brownian motion for which we can compute a wide range of option prices which can be used as control variates (an extensive range of such B&S price formulas is provided in [BFGLMO99]).

6.2.3 Pricing Options on Variance

Recall the definitions on page 60: an *option on realized variance* was a European payoff on the realized variance, while an *option on variance* was a general integrable payoff measurable with respect to the observation of the variance swap prices.

We are interested in particular in the payoff assembled in example 1.3 in the introduction: standard vanilla options on realized variance,

$$\left(\frac{1}{T}\int_0^T \zeta_s\, ds - K^2\right)^+ \quad \text{and} \quad \left(K^2 - \frac{1}{T}\int_0^T \zeta_s\, ds\right)^+ ,$$

or the respective options on realized volatility,

$$\left(\sqrt{\frac{1}{T}\int_0^T \zeta_s\, ds} - K\right)^+ \quad \text{and} \quad \left(K - \sqrt{\frac{1}{T}\int_0^T \zeta_s\, ds}\right)^+ .$$

All these products can be priced with the methods described above without simulating the stock price. The obvious control variates are the prices of variance swaps or forward started variance swaps. As discussed in chapter 4, these swaps are then also used to hedge the exposure to the variance risk.

Another interesting type of payoffs are *options on variance swaps*, for example a call on a variance swap

$$\left(\frac{V_{T_1}(T_1, T_2)}{T_2 - T_1} - K^2\right)^+ \tag{6.30}$$

where $V_t(T_1, T_2) := V_t(T_2) - V_t(T_1)$ denotes the forward variance swap between T_1 and $T_2 > T_1$. The option settles at time T_1 and pays the positive difference between the price at time T_1 of the forward variance swap and the strike.

The pricing of such payoffs is also straight-forward in the current framework, because at time T_1, the price of the forward variance swap between T_1 and T_2 is simply given as $V_{T_1}(T_1, T_2) := \mathbb{G}(Z_{T_1}; T_2 - T_1)$. See also the discussion in section 7.5.4 and the example computations there.

6.2.4 Pricing European Options on the Stock

When we want to price European options, we also do not need to simulate X itself. This is because by conditioning, equation (6.20) on page 93 shows that

$$X_T = X_T^1 + X_T^2$$

with

$$X_T^1 = \varrho_B \int_0^T \sqrt{\zeta_s}\, dB_s^1 - \frac{1}{2}\varrho_B^2 \int_0^T \zeta_s\, ds$$

and

$$X_T^2 = \left(\sqrt{(1-\varrho_B^2) \int_0^T \zeta_s\, ds} \right) B_t^2 - \frac{1}{2}(1-\varrho_B^2) \int_0^T \zeta_s\, ds$$

where B^2 is independent of (B^1, ζ, θ, m). By conditioning, we can therefore compute the price of a call as

$$\begin{aligned} \mathbb{E}\left[\, (S_T - K)^+ \,\right] &= \mathbb{E}\left[e^{X_T^1} \left(e^{X_T^2} - Ke^{-X_T^1} \right)^+ \right] \\ &= \mathbb{E}\left[e^{X_T^1}\, \mathbb{BS}\left((1-\varrho_B^2) \int_0^t \zeta_s\, ds;\ Ke^{-X_T^1} \right) \right] \end{aligned}$$

where

$$\mathbb{BS}(V,k) := \mathcal{N}(d^+) - k\mathcal{N}(d^-) \tag{6.31}$$

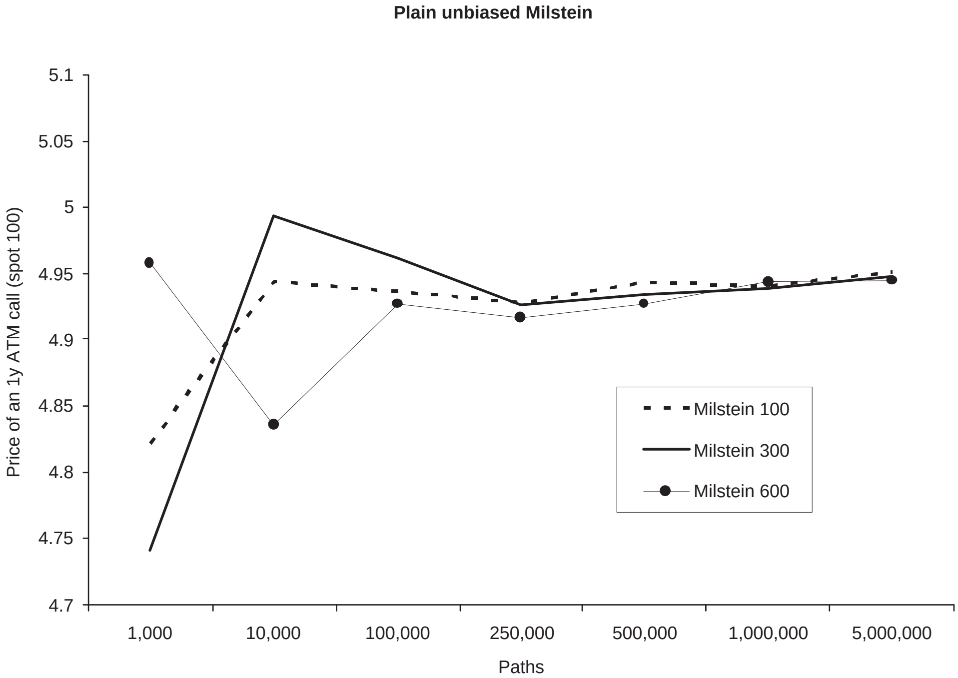
with

$$d^\pm := -\frac{\log(k)}{\sqrt{V}} \pm \frac{1}{2}\sqrt{V}$$

is the Black-Scholes call price function. Hence, to compute the call price in our model, we only need to price the payoff

$$e^{X_T^1}\ \mathbb{BS}\left((1-\varrho_B^2) \int_0^t \zeta_s\, ds;\ Ke^{-X_T^1} \right)$$

This technique reduces the variance of the price considerably (we removed one source of uncertainty). Additionally, we can use control variates for the variance process to ensure quick convergence of the variances. This is shown in figure 6.6 where we compare the unbiased Milstein scheme above with the method described above, where we also used the Black&Scholes calls on the equity as control variates (we used the variance given by the model's variance swap prices).



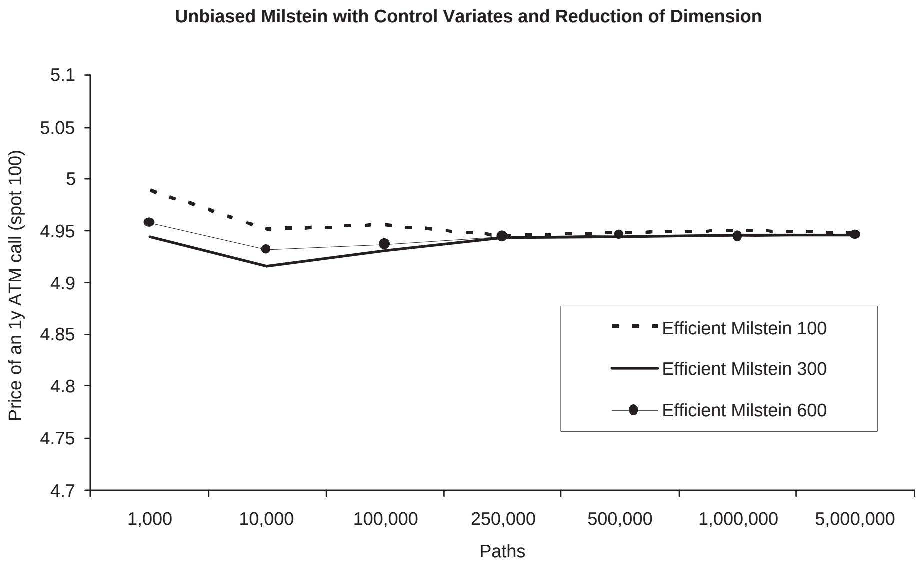


Figure 6.6: Comparison of the plain unbiased Milstein scheme to compute a European call price with the method discussed in section 6.2.4 for various steps per year. In particular, we see that the prices computed with this method are already relatively close to the true value for a very low number of paths.

Chapter 7

Calibration

No model can be used in practise without a reliable and reasonably quick calibration scheme. We therefore describe here how the model (3.7) can be calibrated using the previously discussed Monte-Carlo scheme.

One of the strengths of the model is that it provides two levels of calibration: one initial *full calibration* of all states and all parameters of the model, and a second, much faster *recalibration* of the state parameters only. This second recalibration scheme is used throughout the day or even for longer periods, while the full calibration only need to be executed if the markets move considerably.

7.1 Notation and Overview

First let us clarify the *exotic products* we want to hedge. The term *exotic* is used to indicate that there is no liquid market available to price this product.

Exotic payoffs

We extend the admissible payoff functions of definition 5.1 on page 69 to payoff functions which can also depend on the paths of the prices of variance swaps (this allows us to write options on variance swaps). As before, the key idea is to allow all measurable payoffs which are integrable for all configurations of the model. Recall the definition of the market filtration $\mathbb{F}^{S,V} = (\mathcal{F}_t^{S,V})_{t\geq 0}$ defined in (4.14) on page 60. As in chapter 5, we will use superscripts to denote the dependency of model quantities on the configuration $\xi = (Z_0, \chi)$ with initial states $Z_0 = (\zeta_0, \theta_0, m_0)$ and parameters $\chi \in \Upsilon$.

Definition 7.1 (Exotic payoff) *An* exotic payoff function $\mathbb{H}$ *with maturity T is a measurable non-negative function*

$$\mathbb{H} : C_+[0,T] \times C_+[0,T_1^H] \times \cdots \times C_+[0,T_{d_H}^H] \longrightarrow \mathbb{R}_{>0}$$

for some reference maturities $0 < T_1^H < \cdots < T_{d_H}^H < \infty$, such that for each $\xi = (Z_0, \chi)$, the terminal payoff

$$H_T^\xi(\omega) = \mathbb{H}\left(\left(S_t^\xi(\omega); V_t^\xi(T_1^H)(\omega), \ldots, V_t^\xi(T_{d_H}^H)(\omega)\right)_{t:t\in[0,T]}\right) \tag{7.1}$$

is integrable, i.e. $H_T^\xi \in L^1_+(\mathcal{F}_T^{S,V})$ *and its value at time* t *can be written as*

$$H_t^\xi(\omega) = h^\chi\Big(t; S_t(\omega), Z_t(\omega)\Big)$$

in terms of a continuous function h *which is at least once differentiable in* S *and* Z*.*

PROPOSITION 7.2 *The function* $\mathbb{G}$ *defined in (6.7) is* τ*-invertible. More precisely, for all exotic payoff functions* $\mathbb{H}$ *and all* $0 < T_1 < T_2 < T_3$*, the value* H_t^ξ *of* $\mathbb{H}$ *at time* t *can be written as*

$$H_t^\xi = \hat{h}^\chi\Big(t, V_t^\xi(t)(\omega)\ ;\ S_t^\xi(\omega), V_t^\xi(T_1)(\omega), \dots, V_t^\xi(T_3)(\omega)\Big)$$

and H *can locally be replicated as*

$$H_t^\chi = H_{t_0}^\chi + \int_{t_0}^t \Delta_t^\xi\, dS_t^\xi + \sum_{j=1}^3 \int_{t_0}^t \nu_t^{\xi,j}\, dV_t^\xi(T_j)$$

$t \in [t_0, t_0 + T_1)$ *with the hedging ratios*

$$\Delta_t^\xi := \partial_S \hat{h}^\chi(\cdots) \quad \text{and} \quad \nu_t^{\xi,j} := \partial_{V(T_j)} \hat{h}^\chi(\cdots)\ .$$

Proof – Corollary 4.21 on page 63. □

As a consequence, it is sufficient to hedge our exotic product with only three variance swaps and the stock. The selection of these variance swaps will depend on the product: we assume that each exotic payoff has up to three *pillar dates* which are "important" for the product.

EXAMPLE 7.3 *For a call on realized variance,*

$$\left(\frac{1}{T}\int_0^T \zeta_t\, dt - K^2\right)^+ \ ,$$

the natural pillar date is T*.*

EXAMPLE 7.4 *A* forward variance swap *pays out the realized variance between two futures* $0 < T_1 < T_2$*. We denote by* $V_t(T_1, T_2)$ *its value at any time* t *(see also (6.30) above).*

A call on a variance swap (cf. example 1.3 on page 12) has the payoff

$$\left(\frac{V_{T_1}(T_1, T_2)}{T_2 - T_1} - K^2\right)^+$$

Since

$$V_t(T_1, T_2) = V_t(T_2) - V_t(T_1)\ ,$$

T_1 *and* T_2 *are natural pillar dates for an option on a variance swap.*

Primary and Secondary Market Information

Let us assume that we observe the spot price S_0 of the underlying and a series of variance swap market prices $\mathcal{V} = (\mathcal{V}(T_k))_k$ for $k = 1, \dots, d_V$ with $0 < T_1 < \cdots < T_{d_V}$. We regard these prices as *primary information* to which the theory developed in section 4.2 applies. We will use the entire strip of variance swaps to calibrate the initial states

$$Z_0 = (\zeta_0, \theta_0, m_0)$$

and, possibly, the two reversion-speeds (κ, c). Moreover, the *pillar variance swaps* (those variance swaps which have as maturity one of the pillar dates of the product) will be used to instantaneously re-calibrate the state Z_0 from the market any time later. This ensures that the pillar variance swaps are always well marked, even if the market has moved. It also allows us to compute "VarSwapDeltas", i.e. hedging ratios with respect to variance swaps: the derivative of the price of the exotic payoff is numerically approximated by computing the central difference of the option price if the variance swap price is bumped upward and downward.

Next to stock and variance swaps, we also have *secondary information* in the form of traded European option prices $\mathcal{C} := (\mathcal{C}_\ell)_\ell$ with maturities $\tau_\ell > 0$ and strikes $K_\ell > 0$ for $\ell = 1, \dots, d_c$.[1] We assume $(K_\ell, \tau_\ell) \neq (K_r, \tau_r)$ for $r \neq \ell$ and also that the Black&Scholes *Variance-Vega*[2] of each option is above, say, 0.01.[3] Finally, we assume that for each date $\tau \in \{\tau_1, \dots, \tau_{d_C}\}$, the price of an ATM option is provided, i.e. there is an $\ell \in \{1, \dots, d_C\}$ such that $(K_\ell, \tau_\ell) = (1, \tau)$.

The distinction between *primary* and *secondary* information stems from the underlying assumption of our model: we have assumed throughout that the variance swaps along with the stock are the basic traded contracts which we want to use to hedge our exotic products. The European options are "just" used as a provider of information on the unknown parameters (6.2)

$$\chi = (\kappa, c; \mu, \nu, \eta; \alpha, \beta, \gamma; \bar{m}; \rho_v, \rho_m, \rho_\theta, \rho_{v,\theta}) \ .$$

of the model.

This is not as superficial as it may sound: after all, the information contained in the prices of European options is not sufficient to infer dynamical properties of a model: all information of the surface is fully captured by an implied local volatility model such as Dupire's [D96].[4] Hence, it is necessary to make some structural assumption in our model and to specify which options, or combination of options, are more important for the calibration than others. (In the calibration of classic stochastic volatility models this is often done by allowing the user to specify a "weight" for each option.)

7.2 Numerical Calibration

The calibration of the model is separated into various steps on which we briefly want to comment. The details follow in the next sections.

Phase 1: Market Data Preprocessing

The very first part of a calibration procedure should be a preliminary check of the input market data.

We have assumed that we are given distinct quotes $\mathcal{V} = (\mathcal{V}(T_k))_{k=1,\dots,d_V}$ of variance swap prices and also a set $\mathcal{C} = (\mathcal{C}_\ell)_{\ell=1,\dots,d_C}$ of European call prices. Both price information are typically slightly inaccurate for several reasons: quotes stem from different points in time, bid/ask spreads are aggregated, illiquid option prices might be mismarked or there are plain errors in the data.

[1] As outlined in appendix 7, these assumptions are equivalent to assuming deterministic interest rates and a deterministic forward curve of the underlying with proportional dividends.

[2] *Variance-Vega* is the derivative of the Black&Scholes call price function with respect to its variance parameter (i.e. the derivative of $\mathbb{BS}$ in (6.31) with respect to V).

[3] This condition ensures that the option is sufficiently sensitive to changes in the variance.

[4] See also the discussion at the end of section 2.2.3 on fitting versus structural.

As a result, the input data itself may actually exhibit what we will call *static arbitrage-opportunities*: possible combinations of options that provide a riskless profit. A typical example is a *butterfly* with a negative price. A butterfly has the “hat” payoff

$$\frac{(S_T-(K-k))^+ - 2\,(S_T-K)^+ + (S_T-(K+k))^+}{k^2}$$

for $k \ll K$. It is the second finite difference of the call price function $K \mapsto (S_T - K)^+$; since this function is convex, the butterfly is always positive. If its price is negative or zero, it is therefore possible to enter a *static* position of options for zero or negative cost which has no probability of a loss, but a non-zero probability of making a profit.

Definition 7.5 (Strict absence of arbitrage) *We call a set of market prices* strictly arbitrage-free *if there exists a non-negative martingale S on some stochastic base $(\Omega, \mathcal{F}_\infty, \mathbb{F}, \mathbb{P})$ such that for each product, the market price equals the expectation of the payoff on S under $\mathbb{P}$.*

In that case we say that “S reprices the market”.

Since our input data are likely to be slightly erroneous, we can assume that if we encounter a set of market prices which is not strictly arbitrage-free, the situation is a result of bad data rather than an actual trading opportunity.

But now consider what we are actually going to do: we are going to try to find some model parameters of an *arbitrage-free* model such that they fit as well as possible to the *not* arbitrage-free prices. Hence, there is in inherent mismatch between the model prices and the market data which has nothing to do with the quality of the choice of parameters. Nonetheless, the numerical minimizer which is used to find the market parameters (see below) will still try to minimize this intractable difference. At best, it will just waste some time. At worst, it will abandon an otherwise better fit in search for a reduction of an error which cannot be removed. In all cases, a meaningless objective value as a measure of the quality of the fit will be reported.

All in all, it is necessary to pre-process the market data to ensure that it does not exhibit any inherently impossible configurations. For variance swap prices, this is straight-forward since these prices just need to be increasing in time: in that case the standard Black&Scholes model with the variance set to the variance swaps reprices the market (in our experience, the simplicity of this condition actually means that these data are also much less likely to be wrong).

For the European options, ensuring absence of arbitrage is more complicated. We will discuss an algorithm to detect all possible static arbitrage-conditions and we shall also present a second algorithm which can “correct” arbitragable market data.[5]

Phase 2: Calibration of the state parameters

The next step of the calibration routine is what we will call the *state calibration*: we will use the variance swap price function $\mathbb{G}$ given in (6.7) to calibrate only the states $Z_0 = (\zeta_0, \theta_0, m_0)$ and possibly also the mean-reversion speeds (κ, c) to the observed variance swap prices (with increased weights for the pillar dates). This is very quick. The same algorithm is used for intra-day recalibration of the states to provide instantaneous hedging ratios for exotic products.

[5]The above discussion applies to calibration problems in general: for example we have found that such pre-processing greatly improves the numerical calibration of an implied local volatility function.

Phase 3: Full parameter calibration

After the states Z_0 and the mean-reversion parameters (κ, c) are determined, a *parameter calibration* of the remaining parameters

$$\tilde{\chi} := (\mu, \nu, \eta; \alpha, \beta, \gamma; \bar{m}; \rho_\zeta, \rho_m, \rho_\theta, \rho_{\zeta,\theta}) \tag{7.2}$$

is performed. It is the slowest part of the process and is executed in three subsequent steps: first, we only calibrate the volatility-coefficients with an assumed zero correlation model to the ATM options. Thereafter, we calibrate $\tilde{\chi}$ in two steps to the full set of European options (one step with a relaxed precision and a second step with the target precision).

7.2.1 Intraday Use

Once we obtained a reliable parameter set χ, the model can be used to price exotic payoffs $\mathbb{H}$. Of course, the market may have moved away from our initial market situation.

Recalibration

This is taken into account by an instant implication of the state variables Z_0 from the pillar date variance swap market prices: we simply run the *state calibration* again (with fixed reversion speeds) with far larger weights on the pillar variance swaps and smaller (but not zero) weights on the remaining swaps. This ensures that the model reprices the pillar variance swaps closely.

VarSwapDelta

Moreover, the same routine can be used to compute greeks with respect to the variance swap prices: we can simply move one of the variance swap prices up and then down by, say, 0.5%, imply new states $Z_0^{\pm}$ and reprice the exotic product. This yields via central differences an approximation of the *VarSwapDelta*

$$\begin{aligned}
\nu^k &:= \partial_{V(T_k)} \hat{\mathbb{H}}_r(\chi; \hat{S}_0, \mathcal{V}_0(T_1^r), \dots, \mathcal{V}_0(T_{m_r}^r)) \\
&\approx \frac{\mathbb{H}_r(\dots, 100.5\%\, \mathcal{V}_0(T_k^r), \dots) - \mathbb{H}_r(\dots, 99.5\%\, \mathcal{V}_0(T_k^r), \dots)}{\mathcal{V}_0(T_k^r)/100}\,.
\end{aligned}$$

Since the pillar variance swaps have a far larger weight than the other swaps, we obtain our hedging ratios mainly in terms of the pillar variance swaps.

Note that parameter-hedging ratios as described in chapter 5 can also be computed using the calibrated model by bumping the various parameters.

Summary: Assumptions and some Notation

Here is a brief summary of the market data we consider:

- We are given a set of variance swap market prices $\mathcal{V} = (\mathcal{V}(T_k))_{k=1,\dots,d_V}$.
- Moreover, we observe a range of European call prices $\mathcal{C} = (\mathcal{C}_\ell)_{\ell=1,\dots,d_C}$ with strikes $\mathcal{K} = \{K_\ell\}_\ell$ and maturities and by $\mathcal{T} = \{\tau_\ell\}_\ell$. We will also write $\mathcal{C}(\tau_\ell, K_\ell) := \mathcal{C}_\ell$.

Let $d_\tau := \#\mathcal{T}$. We denote by $\mathcal{K}^\tau := \{K \in \mathcal{K} : \exists \ell : (K_\ell, \tau_\ell) = (K, \tau)\}$ the option strikes provided for the maturity τ and set $d_\tau := \#\mathcal{K}^\tau$. We also write $\mathcal{K}^\tau = \{K_1^\tau, \dots, K_{d_\tau}^\tau\}$. We also define $K_0^\tau := 0$ and $\mathcal{C}(\tau, 0) := 1$.[6]

We abbreviate $\mathcal{K}_\ell := \mathcal{K}^{\tau_\ell}$. Recall that by assumption $1 \in \mathcal{K}_\ell$ for all ℓ.

We also assume there is a *zero price strike* $K^* \gg \max \mathcal{K}$ for which impose a zero call price value, $\mathcal{C}(\tau, K^*) := 0$. We then define $K_{d_\tau+1}^\tau := K^*$, and set the call price for this strike to zero. We also set $\mathcal{K}_\ell^* := \mathcal{K}_\ell \dot\cup \{0, K^*\}$.

- We want to price an exotic product $\mathbb{H}$ with pillar dates $\{T_1^H, \dots, T_3^H\} \subset \{T_1, \dots, T_{d_V}\}$.

7.3 Phase 1: Market Data Adjustment

In the first phase of the calibration, we will check whether the provided European option prices are strictly arbitrage free according to definition 7.5. We will also discuss an algorithm which can adjust given market data in a way which produces a "close" arbitrage-free set of prices.

The idea is to ensure that a discrete-state discrete-time martingale exists which reprices the market. It is also possible to generate the respective transition densities. This has been presented in [B06a] and we will discuss it in appendix D.

The key is the notion of *Balayage-order*:

7.3.1 The Balayage-Order

DEFINITION 7.6 *The* Balayage-*order between two measures μ and ν is defined as*

$$\mu \preceq \nu \quad \textit{iff} \quad \int f(x)\, \mu(dx) \le \int f(x)\, \nu(dx)$$

for all convex functions f. We then say that ν is more expensive *than μ.*

LEMMA 7.7 *We have $\mu \preceq \nu$ if and only if*

$$\int (x-k)^+\, \mu(dx) \le \int (x-k)^+\, \nu(dx)$$

for all k.

For a proof, see corollary 2.63 in Föllmer/Schied [FS04]. The next theorem is due to Kellerer [K72].

THEOREM 7.8 (Kellerer 1972) *Let $(\mu^t)_{t\in\mathcal{J}}$ be a set of probability measures with expectation 1, where $\mathcal{J} \subseteq \mathbb{R}_{\ge 0}$ is some Borel-set.*

Then, a martingale $S = (S_t)_{t\in\mathcal{J}}$ with marginal distributions μ_t exists if and only if μ is in Balayage-order, that is

$$\mu^t \preceq \mu^u$$

for all $t < u$ with $t, u \in \mathcal{J}$.

Moreover, S can be chosen as Markov process.

Theorem 7.8 will be our main tool. Note that it is stronger than Dupire's [D96] result since it is also applicable to non-continuous martingales.

[6] This implies that S is a true martingale, not a local martingale.

7.3.2 Upper Pricing Measures

The link with theorem 7.8 to the question of strict arbitrage among a discrete set of option prices is simply that a discrete set of option prices is strictly free of arbitrage if and only of for each maturity $\tau \in \mathcal{T}$, there exists a measure μ^τ with

$$\int_0^\infty (x - K_i^\tau)^+ \, \mu^\tau(dx) = \mathcal{C}(\tau, K_i^\tau)$$

for all $i = 1, \ldots, d_\tau$ such that the resulting set $(\mu^\tau)_{\tau \in \mathcal{T}}$ is Balayage order.

The first question is therefore how we can construct a measure μ^τ for just one given maturity.

Note that for any true martingale $\mathbb{E}[\, S_T \,] = 1$. Hence, we can set $K_0^\tau := 0$ and $\mathcal{C}(\tau, 0) := 1$. Define then the first difference of the call prices, which is the call spread between two strikes:

$$\Delta_i \mathcal{C}(\tau) := \frac{\mathcal{C}(\tau, K_{i+1}^\tau) - \mathcal{C}(\tau, K_i^\tau)}{K_{i+1}^\tau - K_i^\tau} \qquad i = 0, \ldots, d_\tau \ . \tag{7.3}$$

(recall that $K_{d_\tau+1}^\tau = K^*$ where K^* was the *zero price strike*).

We also set $\Delta_{d_\tau+1}\mathcal{C}(\tau) := 0$.

Definition 7.9 *We call a measure μ^τ* compatible *at time τ if and only if*

$$\int (x - K)^+ \, \mu^\tau(dx) = \mathcal{C}(\tau, K)$$

for all $K \in \mathcal{K}^\tau$.

Following Föllmer/Schied [FS04] section 7.4, we have:

Proposition 7.10 *A compatible measure μ^τ for τ exists if and only if the following conditions hold*

(a) ***Positivity****: For all $K \in \mathcal{K}_\tau$,*

$$\mathcal{C}(\tau, K) \geq 0 \ . \tag{7.4}$$

(b) ***Monotonicity****: For all $i = 0, \ldots, d_\tau - 1$,*

$$-1 \leq \Delta_i \mathcal{C}(\tau) \leq 0 \ . \tag{7.5}$$

(c) ***Convexity****: For all $i = 1, \ldots, d_\tau - 1$,*

$$\Delta_{i-1}\mathcal{C}(\tau) \leq \Delta_i \mathcal{C}(\tau) \ . \tag{7.6}$$

In that case, we can define the upper pricing measure *for τ by*

$$\mu^\tau(dx) := \sum_{i=0}^{d_\tau+1} \delta_{K_i}(dx) \mu_i^\tau \ . \tag{7.7}$$

with

$$\mu_i^\tau := \begin{cases} 1 + \Delta_0 \mathcal{C}(\tau) & (i = 0) \\ \Delta_i \mathcal{C}(\tau) - \Delta_{i-1}\mathcal{C}(\tau) & (i = 1, \ldots, d_\tau + 1) \end{cases}$$

REMARK 7.11 *Instead of monotonicity, it is actually sufficient to assert* $-1 \le \Delta_0 \mathcal{C}(\tau)$ *and* $\Delta_{d_\tau}\mathcal{C}(\tau) \le 0$ *(since monotonicity of the call prices then follows from (7.6) and the requirement that* $\mathcal{C}(\tau,0) = 1$ *and* $\mathcal{C}(\tau, K^\tau_{d_\tau+1}) = 0$*).*

Also note that the above properties imply that $1 \ge \mathcal{C}(\tau, K) \ge (1-K)^+$.

Proof – We shall construct the requested measure. Let $d := d_\tau$ and also omit the notion of τ at the strikes.

Set

$$\mu(dx) := \sum_{i=0}^{d+1} \mu_i \, \delta_{K_i}(dx) \qquad \mu_i \in [0,1] \ .$$

We have to identify μ_i which sum up to 1 and which render μ compatible in τ. First, let

$$q_i := 1 + \Delta_i \mathcal{C}(\tau) \qquad (i = 0, \dots, d).$$

This is the discrete equivalent of $\mathbb{P}[X_\tau \le K] = 1 + \partial_K \mathcal{C}(\tau, K)$. From equations (7.5) and (7.6) we see that $0 \le q_i \le q_{i+1} \le 1$.

Note that if $\Delta_d \mathcal{C}(\tau) < 0$ (which is the case if the call with the highest initial strike has a non-zero price), then $q_d < 1$. We hence set $q_{d+1} := 1$, i.e. $\Delta_{d+1}\mathcal{C}(\tau) := 0$. That is, there is no probability mass beyond the "zero price strike" K_{d+1}.

Also note that we may have $q_0 > 0$ which reflects a possibility of default (since we construct a positive martingale, zero will be an absorbing state).[7]

Now define

$$\mu_i := q_i - q_{i-1} \qquad (i = 1, \dots, d+1) \tag{7.8}$$

and $\mu_0 := q_0$. All μ_i are then non-negative for $i = 0, \dots, d+1$ and they add up to one.

Now let $i \in \{-1, 0, \dots, d+1\}$. Then,

$$\begin{aligned}
\sum_{j=i+1}^{d+1} K_j \mu_j &= \sum_{j=i+1}^{d+1} K_j \left(q_j - q_{j-1}\right) \\
&= \sum_{j=i+1}^{d+1} K_j \left(\Delta_j \mathcal{C}(\tau) - \Delta_{j-1}\mathcal{C}(\tau)\right) \\
&= -\Delta_i \mathcal{C}(\tau) K_{i+1} + 0 + \sum_{j=i+1}^{d} (K_j - K_{j+1}) \Delta_j \mathcal{C}(\tau) \\
&= -\Delta_i \mathcal{C}(\tau) K_{i+1} - \sum_{j=i+1}^{d} \left(\mathcal{C}(\tau, K_{j+1}) - \mathcal{C}(\tau, K_j)\right) \\
&= -\frac{\mathcal{C}(\tau, K_{i+1}) - \mathcal{C}(\tau, K_i)}{K_{i+1} - K_i} K_{i+1} + \mathcal{C}(\tau, K_{i+1}) + 0 \\
&= -\frac{\mathcal{C}(\tau, K_{i+1}) K_i - \mathcal{C}(\tau, K_i) K_{i+1}}{K_{i+1} - K_i}
\end{aligned}$$

On the other hand,

$$K_i \sum_{j=i+1}^{d+1} \mu_j \;=\; K_i \sum_{j=i+1}^{d+1} (q_j - q_{j-1}) \;=\; K_i (q_{d+1} - q_i)$$

[7] This can be avoided by adding a very low strike $K = \epsilon > 0$ with a call price value equal to intrinsic value.

$$
\begin{aligned}
&= K_i(\Delta_{d+1}\mathcal{C}(\tau) - \Delta_i\mathcal{C}(\tau)) \\
&= -\frac{\mathcal{C}(\tau, K_{i+1}) - \mathcal{C}(\tau, K_i)}{K_{i+1} - K_i} K_i
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbb{E}_\mu\left[(X - K_i)^+\right] &= \sum_{j=i+1}^{d+1} (K_j - K_i)\mu_j \\
&= -\frac{\mathcal{C}(\tau, K_{i+1})K_i - \mathcal{C}(\tau, K_i)K_{i+1}}{K_{i+1} - K_i} + \frac{\mathcal{C}(\tau, K_{i+1}) - \mathcal{C}(\tau, K_i)}{K_{i+1} - K_i} K_i \\
&= \mathcal{C}(\tau, K_i)
\end{aligned}
$$

Thus, the measure μ has expectation 1 (by setting $i = -1$) and reprices the market. □

Remark 7.12 *In the above construction of the measure μ, we used the "zero price strike" $K^* = K_{d+1}$ to account for the fact that a market price $\mathcal{C}(\tau, K^\tau_{d_\tau}) > 0$ leaves much room for possible call prices beyond K_d. As long as integrability and convexity is preserved a compatible measure could technically have an infinite support.*

The name "upper pricing measure" is justified by the following observation:

Proposition 7.13 *Let μ^τ be the upper pricing measure for τ.*
Then μ^τ interpolates the call prices linearly.
Consequently,

(a) The measure μ dominates all compatible measures with support only on $[0, K^*]$ in the Balayage-order.

(b) If ν is any compatible measure, than μ is more expensive for all calls with strikes $K \leq K_{d_\tau}$.

(c) If ν is any measure with $\mathbb{E}_\nu\left[(X - K^\tau_i)^+\right] \leq \mathcal{C}(\tau, K^\tau_i)$ for all $i = 0, \dots, d_\tau + 1$, then μ^τ dominates ν in the Balayage order.

For the proof we will need the notion of the linear interpolation between call prices. To this end, define

$$
\mathcal{C}^*(\tau, K) := \frac{K - K_i}{K_{i+1} - k_i}\, \mathcal{C}(\tau, K_{i+1}) + \frac{K_{i+1} - K}{K_{i+1} - K_i}\, \mathcal{C}(\tau, K_i) \qquad K \in [K_i, K_{i+1}) \tag{7.9}
$$

and $\mathcal{C}^*(\tau, K) := 0$ for $K \geq K^*$.

Proof – First we show that $\mu = \mu^\tau$ interpolates the call prices linearly:

$$
\begin{aligned}
\mathbb{E}_\mu\left[(X - K)^+\right] &= \sum_{j=i+1}^{d+1} (K_j - K)\, \mu_j \\
&= \sum_{j=i+1}^{d+1} \left((K_j - K_i) + (K_i - K)\right)\, \mu_j \\
&= \mathcal{C}(\tau, K_i) + (K_i - K) \sum_{j=i+1}^{d+1} \mu_j
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{C}(\tau, K_i) + \frac{K_i - K}{K_{i+1} - K_i} \left(\mathcal{C}(\tau, K_{i+1}) - \mathcal{C}(\tau, K_i)\right) \\
&= \mathcal{C}^*(\tau, K)
\end{aligned}
$$

Now we prove the last of the three statements in the proposition, which obviously implies the first. Number two is just a simple extension.

Let ν be a compatible measure.

For $K \in \mathcal{K}_\tau$ we have $\mathbb{E}_\nu\left[\,(X-K)^+)\,\right] \leq \mathcal{C}(\tau, K) = \mathbb{E}_\mu\left[\,(X-K)^+\,\right]$. So let $K_i < K < K_{i+1}$ (we omit the explicit notion of τ). By convexity of the call price w.r.t. strike,

$$
\mathbb{E}_\nu\left[\,(X-K)^+\,\right] \leq \mathcal{C}^*(\tau, K) = \mathbb{E}_\mu\left[\,(X-K)^+\,\right]\ . \tag{7.10}
$$

i.e. (7.10) applied to μ is an equality. Hence, μ dominates ν. $\square$

Call price functions

Proposition 7.10 makes it clear that the question whether some measures are in Balayage order is a matter of the relationships between the call prices. We therefore define

Definition 7.14 *A* call price function $c : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ *is a function which can be represented as*

$$
c(K) := \int (x-K)^+\,\nu(dx) \tag{7.11}
$$

for some probability measure ν with expectation 1 and support only on $\mathbb{R}_{\geq 0}$.

For a given probability measure ν with support on $\mathbb{R}_{\geq 0}$, its call price function is accordingly defined by (7.11).

As a generalization of proposition 7.10 is (cf. Föllmer/Schied [FS04], lemma 7.23), we have

Proposition 7.15 *A function $c : \mathbb{R}_{\geq 0} \mapsto [0,1]$ is a call price function iff*

(a) c is positive,

(b) c is decreasing,

(c) c is convex and

(d) $c(0) = 1$, $\lim_{x\uparrow\infty} c(x) = 0$ and $c(x) \geq 1 - x$ for $x \in [0, \delta]$ and some $\delta > 0$.

As a result $c(x) \geq (1-x)^+$ for all x.

Proof – Since c is convex and decreasing, its right-hand derivative c' exists and is right-continuous and non-decreasing. Since $c(x) + x \geq 1$ close to 0, it follows that $c'(0) \geq -1$ and therefore $c'(x) \geq -1$ for $x \in \mathbb{R}_{\geq 0}$. Let $f(x) := 1 + c'(x)$, which is a positive, right-continuous and non-decreasing function. These properties imply the existence of some positive σ-finite measure defined via $\tilde{\nu}[(a,b]] := f(b) - f(a)$ (cf. Aliprantis/Border [AB99] theorem 9.47 pg. 354 or Karatzas/Shreve [KS98] pg. 51).

The properties $\lim_{x\uparrow\infty} c(x) = 0$ and $c(x) \geq 0$ imply that $\lim_{x\uparrow\infty} c'(x) = 0$, hence we obtain that $\nu^* := \tilde{\nu}[(0,\infty)] = \lim_{x\uparrow\infty} -f(0) = 0 - c'(0) \in [0,1]$. The proof is complete by defining

$$
\nu(A) := (1-\nu^*)\,\delta_0(A) + \tilde{\nu}(A)
$$

for $A \in \mathbb{B}[\mathbb{R}_{\geq 0}]$. $\square$

We call one call price function c_2 *more expensive* than another call price function c_1 if and only if $c_2(x) \geq c_1(x)$ for all $x \in \mathbb{R}_{\geq 0}$. Thanks to theorem 7.8 that is equivalent to saying that ν_2 is more expensive than ν_1.

So the upper pricing measure μ^τ is just more expensive than all other measures which are compatible with τ because it is the largest (since linear) convex interpolation between the discrete call prices $\mathcal{C}(\tau, \cdot)|_{\mathcal{K}_\tau}$.

We will now need the lower call price function of two such functions

DEFINITION 7.16 *Let c and e be two call price functions. Then,*

$$c \sqcap e := \sup \{ \; h \; : \; h(x) \leq c(x) \wedge e(x) \text{ and } h \text{ is a call price function.} \} \tag{7.12}$$

is called the lower call price function *of c and e.*

For two measures μ and ν with call price functions c and e, we accordingly call the measure $\mu \sqcap \nu$ implied by $c \sqcap e$ the lower *measure of μ and ν.*

We have that $c \sqcap e(0) = 1$ and that $c \sqcap e$ is positive and convex (because the supremum of convex functions is convex). It is decreasing by the properties of the supremum. Hence the above definition makes sense because of the fact that $(1-x)^+ \leq c(x) \wedge e(x)$ ensures that the set on the right hand side of (7.12) is not empty.

Also observe that $\mu \sqcap \nu \preceq \mu$.

DEFINITION 7.17 *We call a set $c = (c^j)_{j=1,\dots,d_\tau}$ of call price functions* strictly arbitrage-free *if the implied measures are strictly arbitrage-free.*

NOTATION 2 *We call $x \in \mathbb{R}_{>0}$ an* extremal point *of c if*

$$\frac{c(x+\delta) - c(x)}{\delta} - \frac{c(x) - c(x-\delta)}{\delta} = \frac{c(x+\delta) - 2c(x) + c(x-\delta)}{\delta} > 0$$

for Lebesgue-almost all $\delta > 0$.

In case c is piecewise linear (as it will be in most of our applications), the extremal points are exactly those points where the slope of c changes.

7.3.3 Relative Upper Pricing Measures

The previous section showed that we can define an upper pricing measure for each maturity under the conditions of proposition 7.10. However, this did not take into account the term structure of the data we have.

For this reason, assume now that we have only two maturities τ_1 and $\tau_2 > \tau_1$ with call strikes $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Recall that $\mathcal{K}_\ell^* := \mathcal{K}_\ell \dot\cup \{0, K^*\}$. We assume that proposition 7.10 applies and that we can construct two upper pricing measures μ_1 and μ_2.

PROPOSITION 7.18 *If $\mathcal{K}_1^* \supseteq \mathcal{K}_2^*$, then the measures μ_1 and μ_2 are in Balayage-order (i.e., $\mathcal{C}$ is arbitrage-free) if and only if $\mathcal{C}(\tau_1, K) \leq \mathcal{C}(\tau_2, K)$ for all $K \in \mathcal{K}_2$.*

Proof – Apply third statement of proposition 7.13. □

In this situation, since the calls of the later maturity must dominate the prices for the earlier, we have to fit the convex τ_1-call price function below those later prices.

Now assume that $\mathcal{K}_1^* \subset \mathcal{K}_2^*$. In this case, the situation is more involved: The upper pricing measure μ_1 might be too expensive between strikes K_{i-1}^2, K_{i+1}^2 for τ_2. The solution is the construction of *relative upper call price measures*:

DEFINITION 7.19 *Let $\mu = (\mu^j)_{j=1,\dots,d}$ with $d \equiv d_\tau$ be the set of upper pricing measures. The set of relative upper call price measures $\bar{\mu} = (\bar{\mu}^j)_{j=1,\dots,d}$ is then given by*

$$\bar{\mu}^d \quad := \quad \mu^d \tag{7.13}$$

$$\bar{\mu}^j \quad := \quad \mu^j \sqcap \bar{\mu}^{j+1} \qquad (j = d-1, \dots, 1) \; . \tag{7.14}$$

By construction, the set $\bar{\mu}$ is increasing with respect to the Balayage order, and each measure μ has support on $\bar{\mathcal{K}}_j := \bigcup_{i=j}^d \mathcal{K}_i^*$.

LEMMA 7.20 *The market is strictly arbitrage-free if and only if the relative upper pricing measures reprice the market.*

PROPOSITION 7.21 *The relative upper pricing measures dominate the marginals of any martingale which reprices the market and whose marginals have support in $[0, K^*]$. For any martingale which reprices the market, the relative upper pricing measure is more expensive for all calls with strikes $K \le K_{d_\tau}^\tau$ for $\tau \in \mathcal{T}$.*

By theorem 7.8, there exists a Markov-martingale S with marginals $\bar{\mu}$, which we will call an "expensive martingale". (Note that it is not unique). It is shown in appendix D how such a martingale can actually be constructed.

Proof of lemma 7.20– First of all, if the relative upper pricing measures reprice the market, then the market is strictly arbitrage-free since $\bar{\mu}$ is in Balayage order.

Conversely, let $j := \max\{j : \bar{\mu}^j$ does not reprice the market$\}$. Then $j < d$. Set $T := \tau_j$, and denote the call price function of μ^T by c and the call price function of $\bar{\mu}^j$ by $\bar{c}$. Since the normal upper pricing measure μ^j reprices the market and since $\mu^j \succeq \bar{\mu}^j$, there exists $K \in \mathcal{K}_j^*$ such that $c(K) > \bar{c}(K)$. We have to show that this yields an arbitrage opportunity.

(a) First assume that $K \in \mathcal{K}_j^* \cap \mathcal{K}_{j+1}^*$.

Since $\bar{\mu}^{j+1}$ reprices the market, we have $\bar{c}^{j+1}(K) = \mathcal{C}(\tau^{j+1}, K)$ (where we denote by $\bar{c}^{j+1}$ the call price function $\bar{\mu}^{j+1}$). Given $\bar{c}(K) = c(K) \sqcap \bar{\mu}^{j+1}(K) < c(K) = \mathcal{C}(\tau^j, k)$ this implies an arbitrage opportunity at K: the call price with strike K for τ_k is more expensive than the price at τ_{k+1}.

(b) So $K \in \mathcal{K}_j^* \cap \bar{\mathcal{K}}_{j+1} \setminus \mathcal{K}_{j+1}^*$.

Define $i := \min\{i > j : K \in \mathcal{K}_i\}$. If $\bar{c}^{j+1}(K) = \mathcal{C}(\tau^i, K)$, we can apply the argument of the point above. If $\bar{c}^{j+1}(K) < \mathcal{C}(\tau^i, K)$, then K is not extremal for $\bar{c}^{j+1}$.

Now let K^- and K^+ be the two extremal points $K^- < K < K^+$ of $\bar{c}^{j+1}$ which are closest to K (note that $\bar{c}^{j+1}$ is piecewise linear, so $K^\pm$ are well-defined). Then, $\bar{c}^{j+1}|_{[K^-,K^+]}$ is a linear function.

Then there exists $i^-, i^+ > j$ such that $\bar{c}^{j+1}(K^\pm) = \mathcal{C}(\tau^{i^\pm}, k^\pm)$. But any valid call price function for τ^j must have $c(K^\pm) \leq c^{i^\pm}(K^\pm)$, which is not possible if $c(k)$ is above the linear interpolation $\bar{c}^{j+1}|_{[K^-,K^+]}(K)$ between $c^{i^-}(K^-)$ and $c^{i^+}(K^+)$

This ends the proof. □

Proof of proposition 7.21– If a martingale S reprices the market, then the market is strictly arbitrage-free and $\bar{\mu}$ reprices the market, too. An argument similar to the above yields that $\bar{\mu}^\tau$ must then dominate the call prices of X_τ on the entire interval $[0, K^{d_\tau}]$. □

The above discussion yields equivalent conditions to a strict arbitrage-freeness. We can employ the results now to implement two algorithms, the first of which tests whether a surface is arbitrage-free and the second of which produces such an arbitrage-free surface "close" to given market data.

7.3.4 Test for Strict Absence of Arbitrage

Lemma 7.20 shows how to check mathematically whether a given call price surface $\mathcal{C}$ is arbitrage-free. From an implementation point of view, the following steps are to be performed:

(a) Ensure the conditions of proposition 7.10 are satisfied for all maturities $\tau \in \mathcal{T}$. Otherwise, the respective marginal itself is not free of arbitrage.

(b) Construct the upper pricing measure μ^{d_τ} for the last maturity τ^{d_τ}. Let $\bar{\mathcal{K}}_d := \mathcal{K}_d^*$. Define $\bar{\mu}^d := \mu^d$, and let $\bar{c}^d$ be its call price function.

(c) For each j,

  (i) Set $\bar{\mathcal{K}}_j := \mathcal{K}_j^* \cup \bar{\mathcal{K}}_{j+1}$.

  (ii) Let $\bar{c}^j := c^j \sqcap \bar{c}^{j+1}$.
  This can be done by the following algorithm:

    i. Set $f(x) := c^j(x) \wedge \bar{c}^{j+1}(x)$ and denote by $1 = K_0 < \cdots < K_{m+1} = K^*$ be the strikes of $\bar{\mathcal{K}}_j$.

    ii. Define $h^0(x)$ as the line between $(1, 1)$ and $(K^*, 0)$.

    iii. For each strike $K_i, i = 1, \ldots, m$, now check whether $f(K_i) \geq h^{i-1}(K_i)$ and set $h^i := h^{i-1}$ in this case
    If $f(K_i) < h^{i-1}(K_i)$ find the strike K_ℓ with $\ell < i$ such that its left hand side derivative is less than
    $$\frac{f(K_i) - h^{i-1}(K_\ell)}{K_i - K_\ell}$$
    (the left hand side derivative at 0 is -1). Such a strike must exist because h^{i-1} is convex.
    Define the function h^i as h^{i-1} on $[0, K_\ell]$, and as linear interpolation between K_ℓ and K_i and K_i and $K_{m+1} = 1$, respectively.

iv. We obtain $\bar{c}^j := h^m$.

(d) Check if $\bar{c}^j(K) = \mathcal{C}(\tau^j, K)$ for all $K \in \mathcal{K}_j$. If not, there is a arbitrage opportunity "in time".

Note that this algorithm also produces the relative upper pricing measures by means of their call prices.

7.3.5 How to Produce a strictly Arbitrage-free Surface

The above algorithm can also be implemented in a linear programming (LP) framework. To this end, we note that the conditions of proposition 7.10 are all linear conditions on the call prices $\mathcal{C}$.

We will now discuss how this observation can be used to produce an arbitrage-free surface from real life market data. As we discussed above, such data is likely to contain small violations of arbitrage.

Let us define as before the sets

$$\bar{\mathcal{K}}_j := \bigcup_{i=j}^{d_\tau} \mathcal{K}_i^*$$

and set $\delta_j := |\bar{\mathcal{K}}_j| - 2$. Define the vectors $K^j = (K_0^j, \ldots, K_{\delta_j}^j)'$ of strikes from $\bar{\mathcal{K}}_j$ and the weighting functions $w^j := (w_0^j, \ldots, w_{\delta_j}^j)'$ with $w_i^j := 1_{K_i^j \in \mathcal{K}_j}$. The weight w^j is therefore zero if there is no price $\mathcal{C}(\tau_j, K_i^j)$ available from the market for K_i^j with maturity τ_j. Note that the positive weights can be altered according to some user-choice.

Then define

$$s_i^j := \mathcal{C}^*(\tau^j, K_i^j)$$

where $\mathcal{C}^*$ is the linear interpolation as defined in (7.9).

We intend to compute a set $c = (c^j)_{j=1,\ldots,d_\tau}$ of call price functions which is as close as possible to the initial market data, i.e.

$$\text{minimize } ||\vec{c} - \vec{s}||_w \tag{7.15}$$

where $||\cdot||_w$ is given in terms of some norm $||\cdot||$ using

$$||\vec{x}||_w := ||\vec{x}\,'w||$$

(recall that a prime $'$ denotes the transpose of a vector). Here, we wrote $\vec{c} = (c_1^1, \ldots, c_{\delta_1}^1, \ldots, c_1^{d_\tau}, \ldots, c_{\delta_{d_\tau}}^{d_\tau})$ and similarly for s.

Clearly, we have to formulate conditions which constrain the minimization problem (7.15) to arbitrage-free call price functions c.

Ensuring absence of strict arbitrage in strike

Now fix some j and define the ratio

$$\alpha_i^j := \frac{1}{K_{i+1}^j - K_i^j}$$

for $i = 0, \ldots, d_j$. (When implementing this algorithm, we have to ensure that the strikes are sufficiently distant from each other to avoid numerical problems.)

In the light of remark 7.11, the conditions of proposition 7.10 translate into

(a) Bounded parameters $1 = c_0^j \geq c_i^j \geq c_{d_j+1}^j = 0$ for $i = 1, \ldots, \delta_j$.

(b) Bounded first derivatives at the boundaries,

$$-1 \leq \alpha_0^j c_1^j + \alpha_0^j c_0^j \quad \text{and} \quad \alpha_{d_j}^j c_{d_j+1}^j + \alpha_{d_j}^j c_{d_j}^j \leq 0$$

(c) Convexity: For $i = 2, \ldots, \delta_j$:

$$\alpha_{i-1}^j c_i^j + \alpha_{i-1}^j c_{i-1}^j \leq \alpha_i^j c_{i+1}^j + \alpha_i^j c_i^j \ .$$

This can also be rewritten as the usual convexity condition

$$\alpha_i^j c_{i+1}^j - \left(\alpha_{i-1}^j + \alpha_i^j\right) c_i^j + \alpha_{i-1}^j c_{i-1}^j \geq 0 \ .$$

Remark 7.22 *In a similar approach to Härdle et al. [FHM03], we can reformulate the above conditions in terms of the first derivatives, too:*

$$\beta_i^j := \alpha_i^j c_{i+1}^j + \alpha_i^j c_i^j \ .$$

In any event all the above conditions are simple linear constraints on the call prices, which we can write as

$$A^j c^j \geq b^j \tag{7.16}$$

for a suitable matrix A^j and a vector b^j.

Strict arbitrage in time

Given now the matrices A^j, we also have to impose the condition that the call prices must be ordered in the Balayage-order. However, since the function c^{j+1} will be defined on all strikes on which c^j is defined, proposition 7.18 yields that it is sufficient to ensure that c^j is below the linear interpolation of c^{j+1}. Because of the convexity conditions on c^j, this is automatically satisfied if $c^j(K_i^{j+1}) \leq c^{j+1}(K_i^{j+1})$ for all $K_i^{j+1} \in \bar{\mathcal{K}}_{j+1} \subset \bar{\mathcal{K}}_j$.

Hence we find a (very sparse) matrix B^j such that

$$B^j \begin{pmatrix} c^j \\ c^{j+1} \end{pmatrix} \geq 0 \tag{7.17}$$

ensures that the call prices are increasing for all $j = 1, \ldots, d_\tau - 1$.

Linear programming

In summary, we have found that the call price vector $\vec{c}$ must satisfy some linear constraints

$$U\vec{c} \geq v$$

to ensure that the resulting call price functions c are strictly arbitrage-free. This can now be used to compute a "closest" fit to the given market data by solving the program

$$\begin{aligned} \text{minimize } &||\vec{c} - \vec{s}||_w \\ U\vec{c} &\geq v \end{aligned} \tag{7.18}$$

Note that this program will return the initial call prices $\mathcal{C}^*$ if the market is strictly arbitrage-free from the start.

REMARK 7.23 *The above "full" linear program can be very extensive, if many maturities are involved. In this case, the program can also be executed "blockwise" from the back, bootstrapping the solution.*[8] *This will not yield a true $||\cdot||_w$-optimal result but is considerably faster.*

REMARK 7.24 *If the sets $\mathcal{K}_j$ are very different from each other, many weights w_i^j will be zero and there is no unique solution to (7.18), which can produce unstable solutions.*

As a remedy, the weight of an artificial call price can be set to some $\varepsilon > 0$.

Note, however, that in our experience this routine does not generally yield an appropriate interpolation if the initial market data exhibits strong arbitrage. The resulting call price surface can be very different from a user's expectation and additional steps such as proper weighting must be taken to ensure that the surface meets the desired properties (such as a tight fit around at-the-money, for example).

7.3.6 Summary

In summary, the algorithm described in the last section is a valuable tool to preprocess the available market data.

For index market data, we have found that the simpler algorithm mentioned in remark 7.23 is sufficient and by far faster (at least if used with a standard LP algorithm such as NAG's `nag_opt_lsq_no_deriv`, see the product documentation [NAG7]). We therefore use this algorithm.

Discrete State Markov processes

The above algorithm yields statically arbitrage-free relative upper pricing measures. Hence, a Markov process S must exist which has these marginal distributions. In [B06a] we have presented an algorithm which is capable of finding transition matrices between the marginal distributions such that the resulting Markov process has indeed the desired marginal distributions. This is lined out in appendix D. The algorithm also takes into account assumed prices of forward-started vanilla options.

7.4 Phase 2: State Calibration

After we have ensured that our call prices form an arbitrage-free set, the next step of the calibration is to infer Z_0 and (κ, c) from the variance swap market prices. To this end, we assume that we are given variance swap weights $w = (w_k)_{k=1,\dots,d_V}$ which reflect the importance of the swaps for the global calibration.

If we have only one exotic product to price, then it is natural to use an increased weight of, say, 10 for the pillar variance swaps and a weight of 1 for all other swaps.

We also assume that the weights are normalized, since this allows the comparison of fits across different calibrations.

[8] Executing it from the back gives a tighter fit than executing it from the front: if we start from the front, a high front call price will drive up the entire surface. If we executed it from the back, we have a tight bound on all call prices.

Our first approach is the problem

$$\underset{\{\zeta_0,\theta_0,m_0;\kappa,c\}}{\text{minimize}} \quad : \sum_{k=1}^{d_V} w_k \|\mathbb{G}(Z_0;T_k) - \mathcal{V}(T_k)\|_2^2 \ . \tag{7.19}$$

This is a constrained non-linear optimization problem which can be solved relatively reliably using methods such as NAG's `nag_opt_nlin_lsq` [NAG7] (the reason for the choice of the L^2-norm is that for this norm more efficient algorithms such as "sequential quadratic programming" exist; see also Press et al. [PTVF02]). Since the function G is known analytically, its derivatives are available in the minimization and (7.19) can be solved with virtually no delay.

Note, however, that minimization algorithms will usually only yield a local minimum. If a previous calibration result $(\bar{\zeta}_0, \bar{\theta}_0, \bar{m}_0, \bar{\kappa}, \bar{c})$ is known it is therefore advisable to impose a *penalty* on the objective function to ensure that the new calibration result is not too far away from the initial position without actually improving the calibration result.

Moreover, it should be noted that variance swaps naturally increase in value with increasing time-to-maturity. That implies that if equal weights $w = (w_k)_{k=1,\dots,d_V}$ are used, then the longer-maturity swaps will have a higher impact on the calibration result. To alleviate this, we propose to minimize over the "variance volatility" (compare definition 1.1 on page 11) of the variance swaps instead of their actual fair value.

Calibration Step 1:

$$\underset{\{\zeta_0,\theta_0,m_0;\kappa,c\}}{\text{minimize}} \quad : \sum_{k=1}^{d_V} w_k \left\| \sqrt{\frac{\mathbb{G}(Z_0;T_k)}{T_k}} - \sqrt{\frac{\mathcal{V}(T_k)}{T_k}} \right\|_2^2 + \lambda(Z_0,\kappa,c) \tag{7.20}$$

where

$$\begin{aligned} \lambda(Z_0,\kappa,c) \quad := \quad & w^\zeta(\zeta_0-\bar{\zeta}_0)^2 + w^\theta(\theta_0-\bar{\theta}_0)^2 + w^m(m_0-\bar{m}_0)^2 \\ & + w^\kappa(\kappa-\bar{\kappa})^2 + w^c(c-\bar{c})^2 \end{aligned} \tag{7.21}$$

for some appropriate penalties $w^\zeta, w^\theta, w^m, w^\kappa$ and w^c. In view of the theoretical results in section 5.3, that varying reversion speeds κ and c will impose arbitrage, we suggest in particular using quite large weights $w^\kappa = w^c = 1$ for the two reversion speeds. We do not use weights for the state parameters.

7.5 Phase 3: Parameter Calibration

7.5.1 ATM calibration

In the next step, we use the European option prices to infer the remaining parameters $\tilde{\chi} := (\mu,\nu,\eta;\alpha,\beta,\gamma;\bar{m};\rho_\zeta,\rho_m,\rho_\theta,\rho_{\zeta,\theta})$.

Calibration Step 2:
In principle we once again run a minimization scheme

$$\underset{\tilde{\chi}}{\text{minimize}} \quad : \sum_{\ell=1}^{d_C} w_\ell^c \left\| \mathcal{C}(Z_0;\chi;\tau_\ell,k_\ell) - \mathcal{C}_\ell \right\|_2^2 + \lambda^c(\tilde{\chi}) \tag{7.22}$$

where $\mathcal{C}(Z_0,\chi;\tau,k)$ is the model call price with maturity τ and strike k. The vector $w^c = (w_1^c,\dots,w_{d_C}^c)$ is once again a weight vector. It is suggested to leave it as user input with default values $w^c_\ell = 1/d_C$ for all ℓ. In any event, normalization to $\sum_{\ell=1}^{d_C} w_c = 1$ should be ensured. Finally, λ^c is a quadratic penalty function just as defined in (7.21) above. We suggest penalties for the cross-correlation term $\rho_{\zeta,\theta}$ of 0.1 and modest penalties of 0.001 for all other parameters. Note that due to the structure of the model, the variance swap prices will not change if $\tilde{\chi}$ is altered.

As mentioned before, we have to revert to Monte-Carlo for the computation of the European option prices.

We will use the scheme (6.29) introduced in the previous section 6.2 to simulate the variance processes and we will apply the method discussed in section 6.2.4 to price the European options. We also ensure that the dates $\{\tau_1, \dots, \tau_{d_C}\}$ are among the simulation dates $0 = t_0 < t_1 < \cdots < t_M := T$ where $T := \max_j \tau_j$. We use Black & Scholes European option prices with the variance equal to the variance swap price of the model as control variates for all maturities $\{\tau_1, \dots, \tau_{d_C}\}$.

Remark 7.25 (Implications of Calibration with Numerical Approximations)
If we assume that N paths are used per simulation of the option prices, it means that the resulting parameter vector $\tilde{\chi} =: \tilde{\chi}_N$ does not necessarily minimize (7.22) but the approximate problem

$$\textit{minimize } \tilde{\chi}_N \quad : \quad \sum_{\ell=1}^{d_C} w^c_\ell \, \|\mathcal{C}_N(Z_0; \chi_N; \tau_\ell, k_\ell) - \mathcal{C}_\ell\|_2^2 + \lambda^c(\tilde{\chi}) \tag{7.23}$$

where $\mathcal{C}_N$ denotes the European option price computed with N Monte-Carlo paths. Obviously, there is an inherent error in $\mathcal{C}_N$ as opposed to $\mathcal{C}$, and we cannot expect to obtain a better fit than this error. In particular, it means that using more than N paths in subsequent option pricing does not improve the pricing error of the reference instruments any further.

Managing the Random Number Generation

Following our previous remarks, we have to ensure that the random numbers used in one Monte-Carlo estimations are the same as the number used in subsequent estimations: if the number of steps and the number of paths is fixed, then for each step of each path, the random numbers used in two estimations of the values of the European options should be the same (only the parameters of the model change). This can be achieved by storing all the numbers used in a big vector ahead of the computation.

The number of random variables required is $3MN$ (recall that we do not simulate the Brownian motion B which drives the stock because we make use of section 6.2.4). If stored as `float` numbers, this amounts to $24MN$ bytes of data. For example, if $M = 1500$ (200 steps per year for the first 5 years and 100 steps per year for the next 5 years) and $N = 10000$, this requires 360 mega-bytes of memory, easily available on today's desktop computers.[9]

Note that in a typical implementation of a least-squares minimization routine such as NAG's `nag_opt_lsq_no_deriv` [NAG7], not all the options will be computed in each step of the minimization (for example, during "line search", the minimizer will compute the values of only a single option).

Storing the numbers ahead of the simulation is then helpful since we have easy access to the random numbers needed for the simulations of the paths up to the relevant maturity (by contrast, if a standard random number generated were used, then we would actually have to generate all

[9]If we want to calibrate to very long term options, this might not be feasible. In this case, we recommend to use a weak approximation scheme as discussed on page 97.

the unused numbers to ensure that we always begin each path with the same number if we want to simulate the process on a path after path basis). It is also helpful when implementing a multi-threaded Monte-Carlo path generator (where paths are generated in parallel on different processors).

7.5.2 Calibration in Steps

The calibration will be performed in several steps aimed at speeding up the process. We fix the number of paths N, and the fixings $t_1 < \cdots < t_M$. As usual for minimization methods, the user is also required to provide a *tolerance* $\varepsilon > 0$ which describes the accuracy of the minimization (7.23). We define a lower "trial" number of paths $N_1 := N/2$ and relaxed tolerance of $\varepsilon_1 := \sqrt{10}\varepsilon$.

Given $(\zeta_0, \theta_0, m_0; \kappa, c)$ we perform the following steps:

(a) **ATM Calibration**:
We first solve (7.23) for only the ATM options $\mathcal{C}_\ell$ such that $k_\ell = 1$. We also assume vanishing calibration parameters $\rho_\zeta = \rho_\theta = \rho_m = \rho_{\zeta,\theta} = 0$. We use N_1 paths and a tolerance of ε_1. The resulting parameter set is

$$\tilde{\chi}^1 = (\mu, \nu, \eta; \alpha, \beta, \gamma) \ .$$

The motivation behind this approach is that the ATM option prices depend only very little on the calibration parameters.[10] The starting point for the calibration is provided by the user and is typically the result of a previous calibration.

(b) **Trial Calibration**:
Starting in $\tilde{\chi}^1$, we now solve the full problem (7.23) for all options with N_1 paths and a tolerance of ε_1 and obtain a parameter set

$$\tilde{\chi}^2 = (\mu, \nu, \eta; \alpha, \beta, \gamma; \bar{m}; \rho_\zeta, \rho_\theta, \rho_m, \rho_{\zeta,\theta}) \ .$$

(c) **Final Calibration**:
The last step is to start in $\tilde{\chi}^2$, and solve (7.23) for all options with N paths and the initial tolerance of ε. The resulting parameter set

$$\tilde{\chi}_N = (\mu, \nu, \eta; \alpha, \beta, \gamma; \bar{m}; \rho_\zeta, \rho_\theta, \rho_m, \rho_{\zeta,\theta}) \ .$$

is used as final return value of the calibration.

7.5.3 Examples

In this sub-section we present some example calibrations. We have only used index data, and our main focus was FTSE and STOXX50E data. We have removed all forward and interest information as discussed in appendix A.2.

The routine has been developed in Microsoft Visual C++ .NET 2003 [MSVC7] and uses code optimized for Pentium 4 processors. The numerical minimizer employed is `nag_opt_lsq_no_deriv` from the NAG Build 7 package [NAG7].

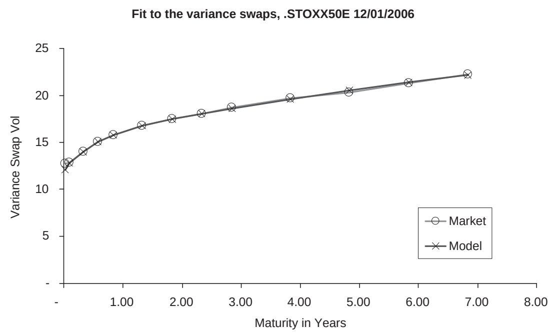
[10] In fact, by our experience it is also worth fixing $\bar{m}$ and γ to the previously calibrated values or values which fit well by experience.

All examples below have been computed on a dual P4 Xeon 4GHz machine with 2GB main memory. The core Monte-Carlo engine is scalable and we have used one thread per available processor.

We have used $N := 7500$ paths for the main routine and 150 steps per year.

Variance Swaps

Each data set contains a sequence of variance swap market prices. These prices are internal quotes, but given the liquidity of the markets in question, these can be seen as market prices. Figure 7.1 shows the fits of the variance curve functional to the market data. Following market conventions, the value of a variance swap $V(T)$ is quoted in its annualized volatility $\sqrt{V(T)/T}$. The parameter values are given in table 7.1.



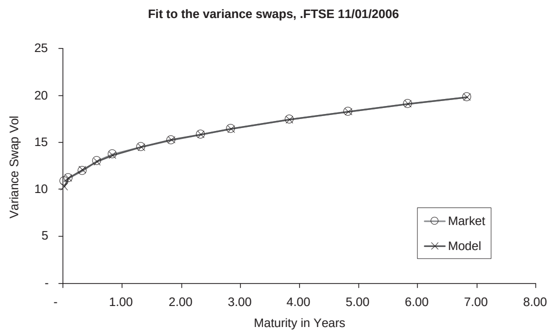


Figure 7.1: Fit of the double-mean reverting functional (6.7) to FTSE and STOXX50E market data. Note that the variance swap prices are quoted according to market standard using "variance swap volatilites", cf. (1.2).

	STOXX50E	FTSE
ζ_0	0.015	0.011
θ_0	0.030	0.021
m_0	0.580	0.145
κ	3.027	3.418
c	0.013	0.058

Table 7.1: Calibration results for STOXX50E and FTSE. The model routinely produces high Short-RevSpeeds κ and comparatively low LongRevSpeeds c, if they are calibrated alongside the state variables.

European Options

In addition to the variance swaps, the model is fitted to a strip of European options. We display in figure 7.2 the fit of the model to a range of options. The calibration results are shown in table 7.2.

We contrast this to the fit of the "reduced" one-factor model from example 3.6 on page 40, where the short variance is given as the solution to the SDE

$$\left.\begin{aligned} d\zeta_t &= \kappa(\theta(t)-\zeta_t)\,dt + \nu\sqrt{\zeta_t}\,dW_t \\ d\theta(t) &= c(m-\theta(t))\,dt\ . \end{aligned}\right\} \tag{7.24}$$

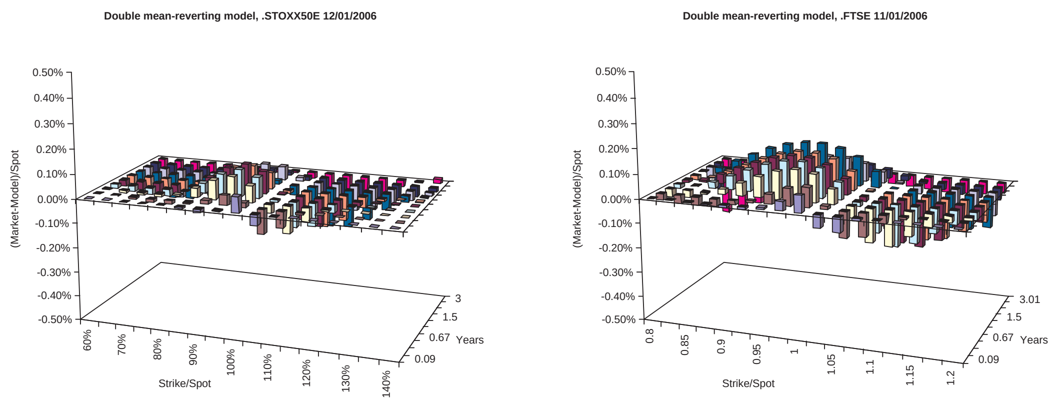


Figure 7.2: Calibration of the double-mean reverting model to STOXX50E and FTSE market data. The graphs show the difference in call prices (divided by spot) for maturities between 3 months to 3 years in a strike range from 80% to 120%. The variance swap fits are shown in figure 6.2.

	STOXX50E	FTSE
ν	0.423	0.431
μ	0.119	0.229
η	0.355	0.363
α	0.500	0.547
β	0.501	0.629
ρ_ζ	-0.68	-0.80
ρ_θ	-0.60	-0.57

Table 7.2: Calibration results for STOXX50E and FTSE. We have also set $\epsilon := 0.0001$, $\bar{m} := 0.0001$, $\rho_{\zeta,\theta} := 0$, $\rho_m := 0$ and $\gamma := 1$.

This model features the same variance curve functional (6.6) as the full model. Since it is essentially Heston's model with time-dependent mean-reversion level, we can compute European options on the equity relatively efficient, cf. Bermudez at al. [BBFJLO06]. Figure 7.3 shows that fitting the remaining free parameters ν and ρ[11] and to European options yields a much worse fit than for the full model. The calibration results themselves can be found in table 7.3.

It should be noted that the fit can be improved considerably by using piece-wise time-dependent correlation and VolOfVol parameters. This is shown in figure 7.4 on page 123.

	STOXX50E	FTSE
ν	0.543	0.820
ρ	-0.90	-0.81

Table 7.3: Calibration results for STOXX50E and FTSE for the reduced model (7.24).

[11] The correlation ρ is the correlation between W and the Brownian motion B driving the associated stock price process.

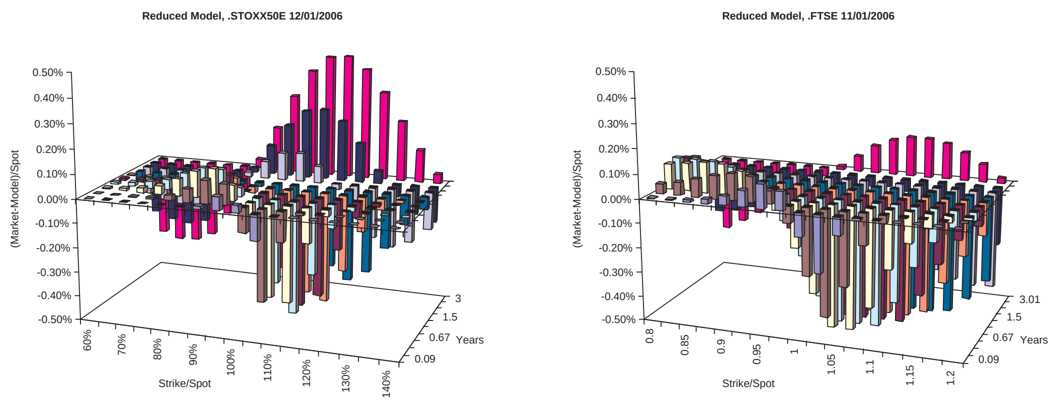


Figure 7.3: Calibration of the reduced model (7.24) to STOXX50E and FTSE market data.

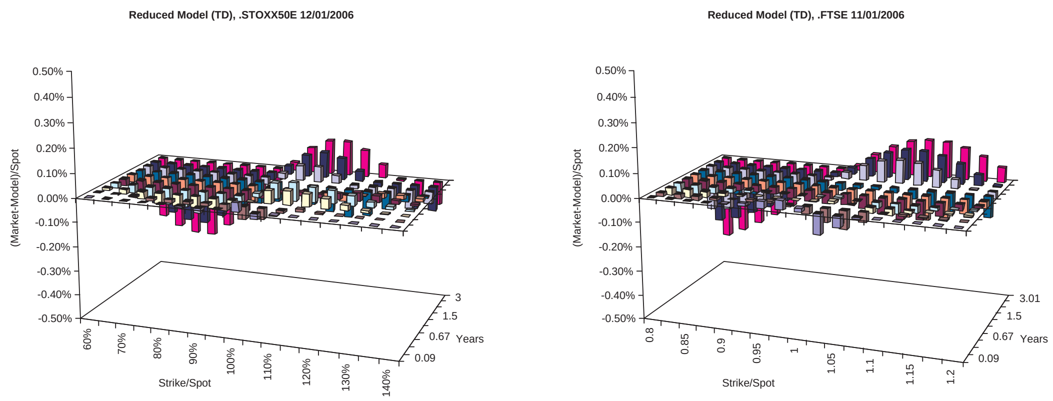


Figure 7.4: Calibration of the reduced model (7.24) with piece-wise constant VolOfVol and correlation parameters to STOXX50E and FTSE market data.

Dynamics of the variance swap curve

Having calibrated both the full and the reduced model, we can now visualize the dynamics of the variance swap curve, as driven by the model over time. For both the full model and the reduced model (7.24) we compute sample paths for

(a) The 3m fixed time-to-maturity variance swap.

(b) Every year from today to 4y, the implied variance swap curve for variance swaps from 60 days to four years.

The results for four sample paths for each model are given in figure 7.5 and figure 7.6, respectively.

In general, the three-factor nature of the full model allows a much richer and more realistic behavior: while the long term level of volatility in the reduced one-factor model is very "sticky", it is allowed to move with the market in the full model: note, in particular, the case of the graph in the lower right corner of figure 7.5 where an upward trend of the level of the 3m fixed time-to-maturity variance swap is accompanied by an overall increase of variance swap prices. Compare this behavior with the top right graph in figure 7.6: while volatility is moving upward, the variance swap curve experiences a reversion of slope and is finally pulled back. This is a typical drawback of one-factor mean-reverting models where the long-term level is fixed.

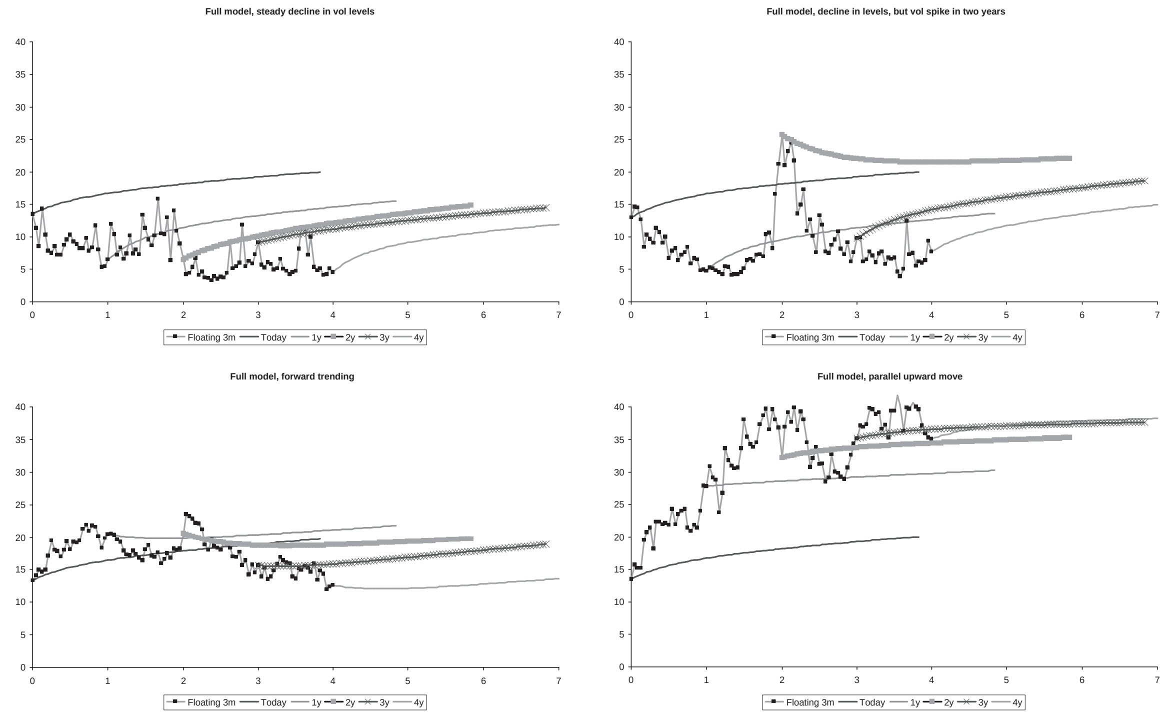


Figure 7.5: Future shapes of the variance swap curve produced by the full model. The model allows for various market scenarios, including down and upward trending volatilities. All quotes are given as "variance swap volatility".

7.5.4 Pricing Options on Variance

In this final section, we now show how the full and the reduced model compare when it comes to pricing options on variance. Define the price of a forward variance swap as

$$V_t(T_1, T_2) := V_t(T_2) - V_t(T_1) = \mathbb{E}\left[\int_{T_1}^{T_2} \zeta_t \, dt \,\middle|\, \mathcal{F}_t \right] .$$

We then priced the following three products, all of them quoted using the market convention to divide the actual payout by twice the square-root of today's variance swap price for the respective period:

- Straight-forward calls on realized variance with payoff

$$\frac{\left(\int_0^T \zeta_t \, dt - k \, V_0(T) \right)^+}{2\sqrt{V_0(T) T}}$$

  for maturities T of three months, six months and one year.

- Forward started calls on realized variance with payoff

$$\frac{\left(\int_{T_1}^{T_2} \zeta_t \, dt - k \, V_{T_1}(T_1, T_2) \right)^+}{2\sqrt{V_0(T_1, T_2)(T_2 - T_1)}}$$

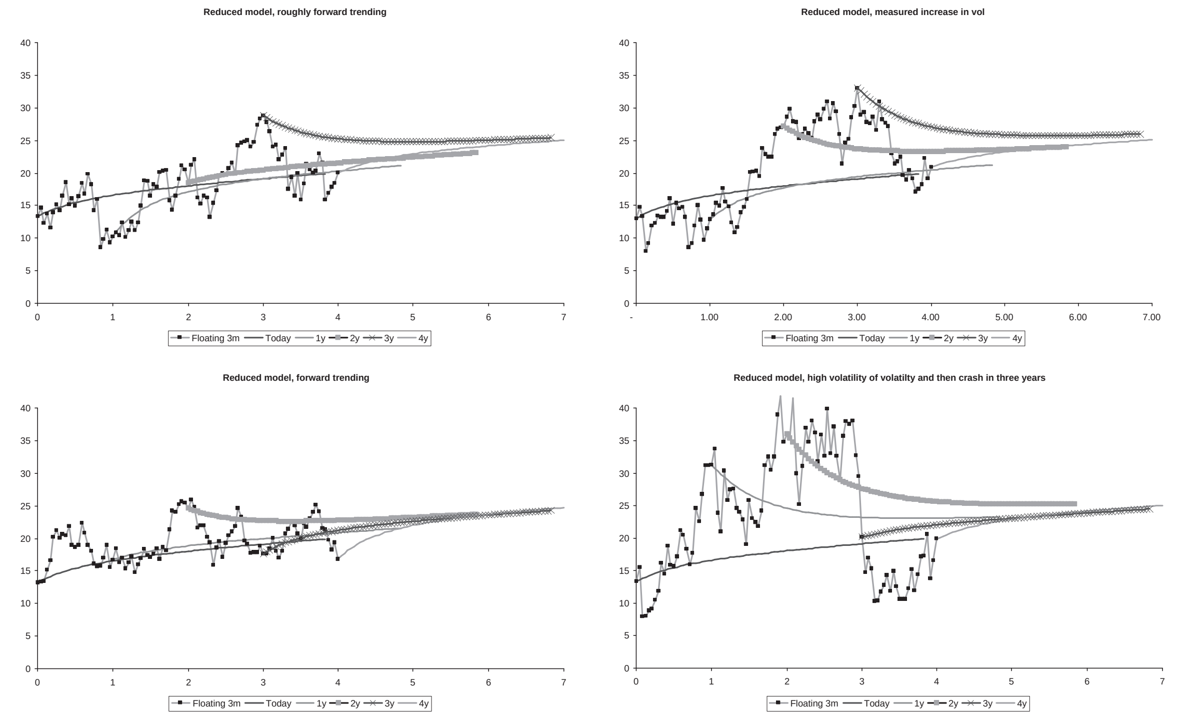


Figure 7.6: Future shapes of the variance swap curve produced by the reduced model. In comparison to figure 7.5, the long-term volatility of the variance swap curves is far more "sticky".

- Options on variance swaps with payoff

$$\frac{\Big(V_{T_1}(T_1,T_2) - k\,V_0(T_1,T_2)\Big)^+}{2\sqrt{V_0(T_1,T_2)(T_2-T_1)}} \tag{7.25}$$

  payable at T_1.

The results are collected in figures 7.8 to 7.11.

Remark 7.26 *In all cases, we have computed the options as payoffs based on realized variance as defined in standard contracts,*[12] *i.e. we have in fact used*

$$\sum_{i=1,\dots,n} \left(\log \frac{S_{t_i}}{S_{t_{i-1}}}\right)^2 \tag{7.26}$$

over business days $0 = t_0 < \cdots < t_n = T$ *instead of the quadratic variation* $\int_0^T \zeta_t\,dt$. *This ensures that the options are correctly priced for short maturities, cf. figure 7.11.*

[12] For examples, see chapter B in the appendix.

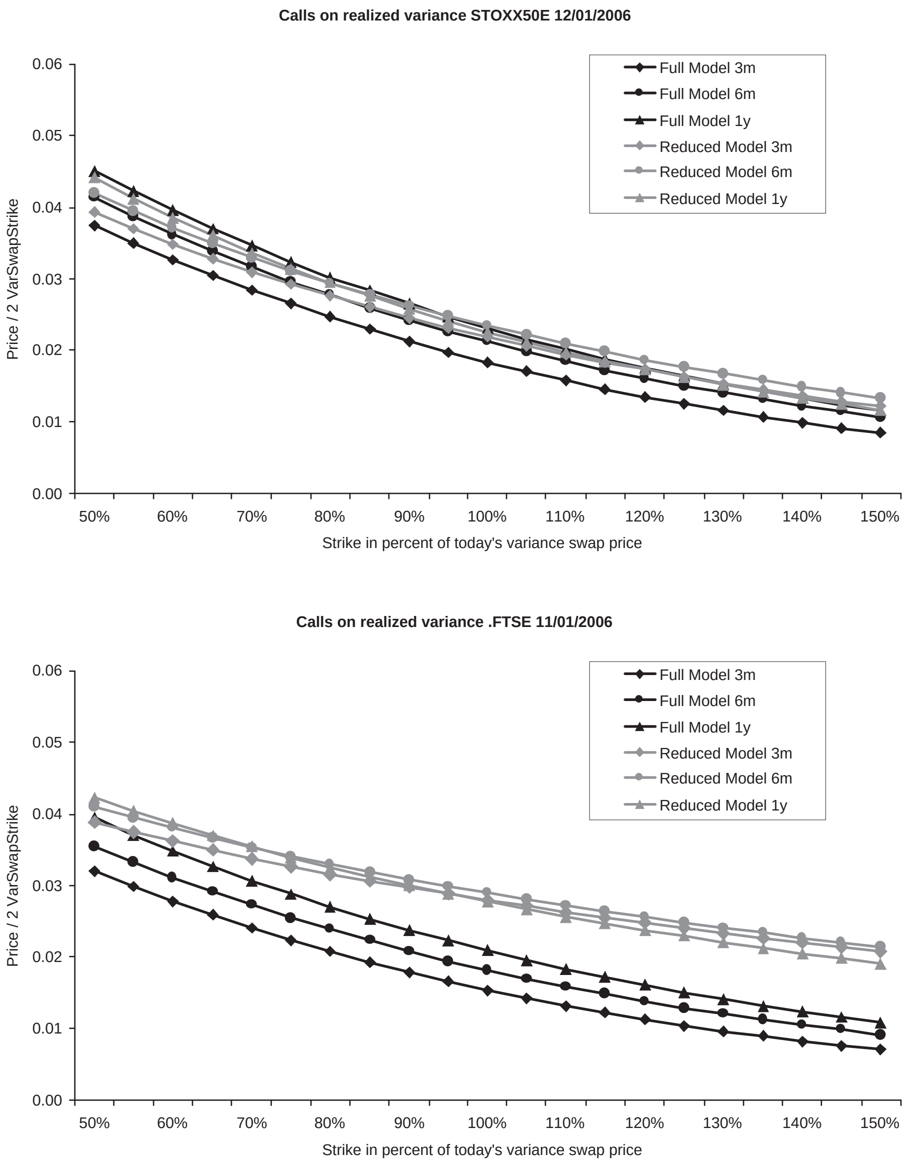


Figure 7.7: Calls on realized variance. A detailed view of the 1y call is given in figure 7.8.

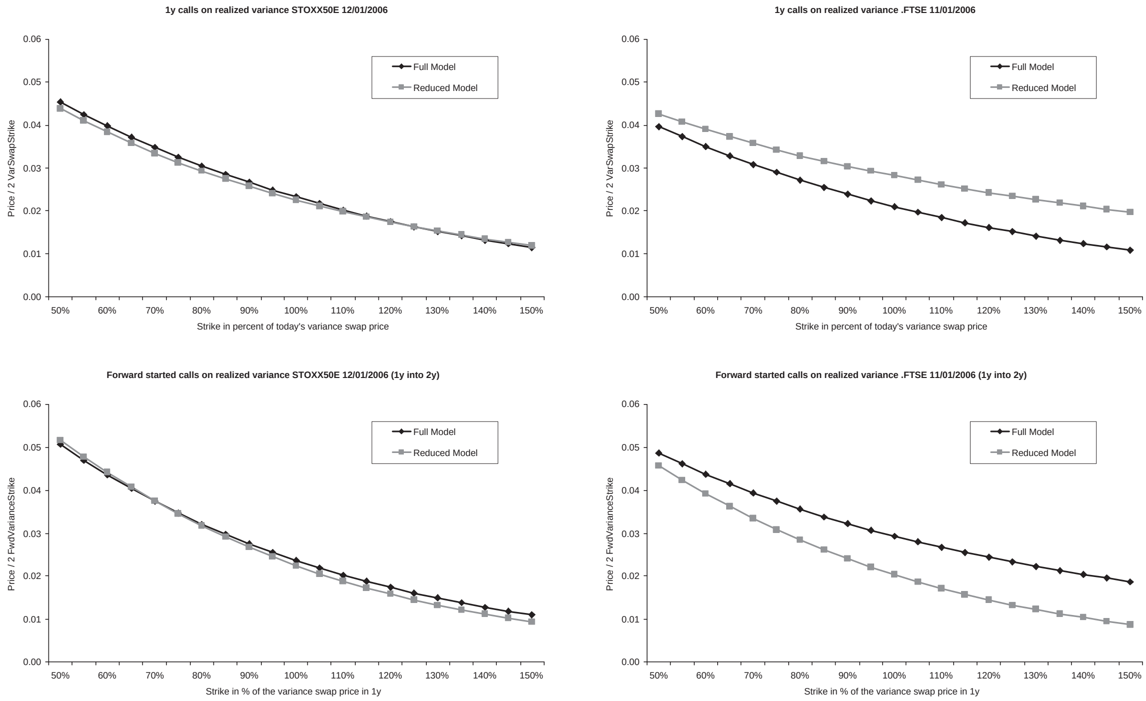


Figure 7.8: Calls on realized variance with one year maturity and forward started calls on realized variance for one year into two years. We find that if the models coincide for the spot started options, then they usually also give similar prices for the forward started version. This is in stark contrast to the prices for options on variance swaps, cf. figure 7.9.

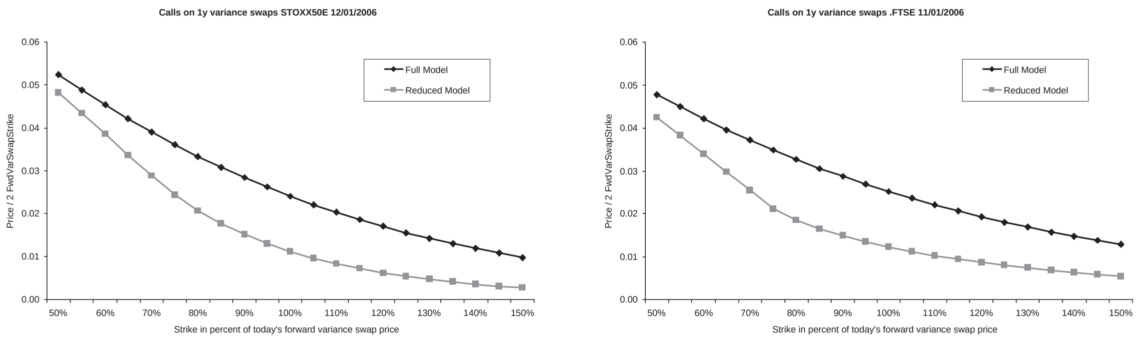


Figure 7.9: Calls on a forward variance swap according to (7.25). Note the difference in the option prices even for STOXX50E, for which the spot started and forward started options are very similar. The difference in prices for options on variance swaps in the two models is due to the fact that the full model allows more variation in the term structure of variance swaps, as can also be seen by comparing figures 7.5 and 7.6. This highlights the importance of the use of multi-factor models for pricing term-structure deals. See also figure 7.10.

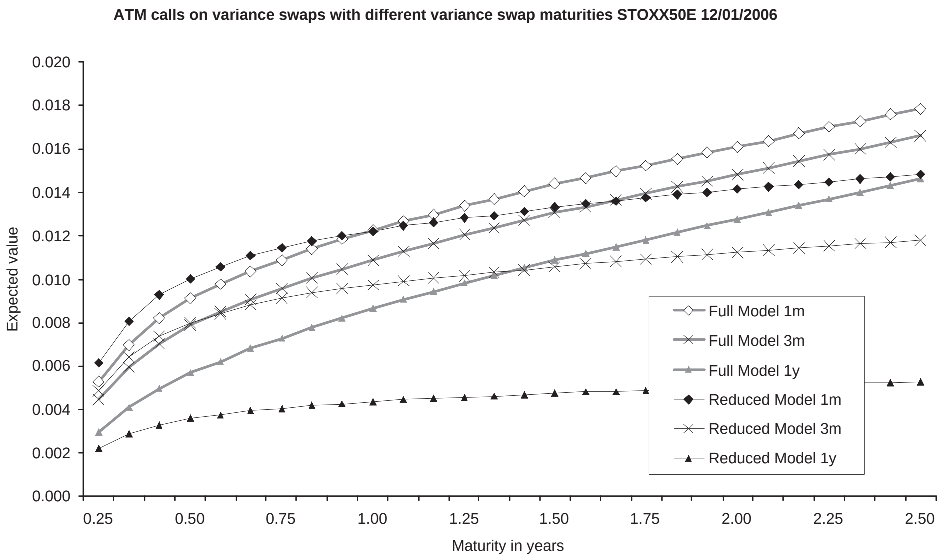


Figure 7.10: Term structures of ATM calls on variance swaps with time-to-maturity of one month, three months and one year (here, we show the expected value of $(V_{T_1}(T_1,T_2) - k\,V_0(T_1,T_2))^+ / (T_2 - T_1)$, rather than the payoff (7.25)). The discrepancy between the full and the reduced model rises as the time-to-maturity of the underlying variance swap increases. Also note the relative decline in time value for the reduced model, which is a consequence of ζ of (7.24) converging to its invariant distribution.

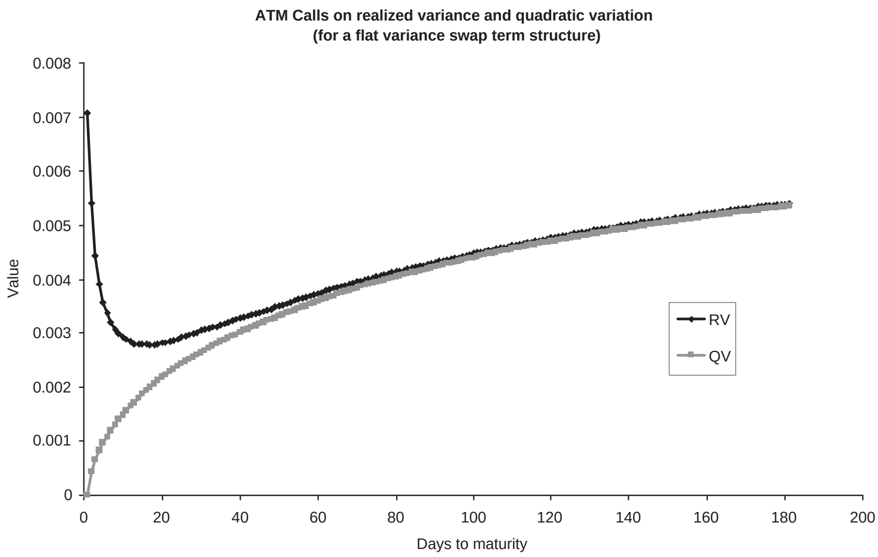


Figure 7.11: ATM calls on realized variance and quadratic variation in a Heston model with flat variance swap term structure. The graph shows that while the approximation of realized variance via quadratic variation works very well for variance swaps, it is not sufficient for non-linear payoffs with short maturities. The effect is common to all variance curve models (or stochastic volatility models, for that matter).

Chapter 8

Conclusions

We have developed a new framework for modeling equity markets by introducing variance curve market models. We have proposed using variance swaps in the same as bonds are used in Heath-Jarrow-Merton interest models and we have shown how finite-dimensionally driven models can be characterized.

We have discussed when general Markov-driven pricing models are complete and we have applied this theory to variance swap curve models. We have also discussed theoretical and practical aspects of "parameter-hedging" which is meant to reduce the risk of a change in a parameter vector.

Finally, we have proposed a particular model. We have shown that this model indeed generates a true martingale process and how an efficient unbiased Monte-Carlo scheme can be developed. We have also shown how this model can be calibrated to real life market data and have commented on the impact of using multi-factor models when pricing options on variance.

Acknowledgements

I am deeply grateful to Prof. Dr. Alexander Schied (Technical University Berlin) for his great help during this project, for his patience and readiness to support me over quite a distance by all means of modern communication, but also by good old face-to-face meetings all over Europe. Without his continuing advice, both mathematically but also personally, this thesis would not have been possible. His admirable deep knowledge of so many different areas of mathematics were a great motivation throughout. To work with him and his team, Mesrop Janunts, Stefan Sturm and Wiebke Wittmüß, was a great pleasure. Indeed, Alexander Schied's mathematical lectures, which I heard first at Humbolt Universität Berlin during my diploma studies, were not only an essential fundament but also an inspiration for my work today.

Moreover, my deepest gratitude goes to Dr.^{2} Marcus Overhaus, head of Deutsche Bank Quantitative Products – this thesis was written in parallel to a full employment in his Quantitative Products Analytics team. His continuing help and trust were essential for the success of this project, and I am very grateful not only for his willingness to support this work, but also for his invaluable guidance and personal advice during the last four and a half years. Marcus Overhaus was greatly supportive by many means, and I am very delighted to have found such a great place like his team to work in the finance industry.

My work would have also not been possible without the help and patience of my colleagues, Dr. Ana Bermudez, Dr. Andy Ferraris (whom I owe much experience, advice and insight since I started my work at Deutsche Bank), Aziz Lamnouar, Dr. Christopher Jordinson (whose relentless strife to improve my command of the English language shall some day bear fruit) and Anwar Puthu. Also, I want to thank Jack Busta who gave me much insight in the reality of finance and the nature of "the market". I am highly indebted for his expertise and willingness to listen to many of my obscure ideas.

I shall also not forget Prof. Dr. Josef Teichmann, Dr. Martin Forde and Holger van Bargen all of whom discussed with me many details of this work, and, of course, Dr. Laurent Nguyen-Ngoc, who, in November 2002, in the Hilton Hotel Paddington Station, asked the right question at the right time.

I also want to thank my old friends who endured my mental and physical absence over the course of the last two and a half years. All of them have been very patient and understanding, and I am grateful for their willingness to keep up with much pressure on our friendships – but I promise to make it up !

Finally, I want to say thank you to my parents, Annegret and Walter, and my sister, Anna, whose love and patience were so important for this project. They gave me the trust and motivation to finish this project and not to lose sight when progress was slow and painful. As for all my life, they have been there and with me, for me, and so gave me trust in myself, for which I am very very grateful.

Thank you.

Appendix A

Variance Swaps, Entropy Swaps and Gamma Swaps

In this appendix, we show how the method introduced by Neuberger [N92] is used to price variance swaps and entropy swaps if a complete set of European options is traded for all strikes and maturities. Standard references on this subject are Demeterfi et al. [DDKZ99] and Carr/Madan [CM02]. Conceptually, we will now follow quite a different approach than before, since we will assume in this section that the European options are the primary liquid instrument and use these to derive the prices of the relevant swaps.

We will still assume that there are no interest rates and no dividends present. In the next section A.2 we then show that this is equivalent to assume deterministic interest rates and repo rates and that the stock pays deterministic proportional dividends.[1] We will also discuss and extension of entropy swaps, called *gamma swaps*, which are easier to explain to clients.

A.1 Basic Pricing and Hedging

The following proposition is central to the argument below:

Proposition A.1 *Let $f : \mathbb{R}_{>0} \to \mathbb{R}$ be a twice differentiable function. Then,*

$$\begin{aligned} f(x) - f(x_0) &= f'(x_0)\,(x - x_0) \\ &+ \int_0^{x_0} f''(k)\,(k-x)^+ \; dk + \int_{x_0}^{\infty} f''(k)\,(x-k)^+ \; dk \; . \end{aligned}$$

for $x, x_0 \in \mathbb{R}_{>0}$.

Proof – Fundamental calculus shows

$$\begin{aligned} f(x) - f(x_0) &= \int_{x_0}^{x} f'(y)\,dy \\ &= \int_{x_0}^{x} \Big(f'(x_0) + \int_{x_0}^{y} f''(k) dk \Big)\; dy \\ &= f'(x_0)(x - x_0) \end{aligned}$$

[1] In Bermudez/Buehler/Ferraris/Jordinson/Overhaus/Lamnouar [BBFJLO06], the impact of discrete cash dividends is explored.

$$
\begin{aligned}
& +1_{x\geq x_0}\int_{x_0}^{x}\int_{x_0}^{y} f''(k)\,dk\,dy + 1_{x<x_0}\int_{x}^{x_0}\int_{y}^{x_0} f''(z)\,dk\,dk \\
= \;& f'(x_0)(x-x_0) \\
& +1_{x\geq x_0}\int_{x_0}^{x} f''(k)\int_{k}^{x} dy\,dk + 1_{x<x_0}\int_{x}^{x_0} f''(z)\int_{x}^{k} dy\,dk \\
= \;& f'(x_0)(x-x_0) \\
& +1_{x\geq x_0}\int_{x_0}^{x} f''(k)(x-z)\,dk + 1_{x<x_0}\int_{x}^{x_0} f''(k)(k-x)\,dk \\
= \;& f'(x_0)(x-x_0) \\
& +\int_{x_0}^{\infty} f''(k)(x-k)^+\,dk + \int_{0}^{x_0} f''(k)(k-x)^+\,dk
\end{aligned}
$$

□

Neuberger's [N92] formula to compute the price of a variance swap is a direct application of the previous proposition (see also Demeterfi et al. [DDKZ99] for a practical discussion of hedging issues and implementation).

ASSUMPTION 6 *The standing assumption in this section is that S is a true continuous martingale and that European option prices are traded for all strikes at the relevant maturities. As before, we assume zero interest rates.*

We denote by ζ the short variance of S (cf. proposition 2.2 on page 20). The time-t price of a call with strike K and maturity T is denoted by

$$\mathcal{C}_t(T,K) := \mathbb{E}\left[\, (S_t - K)^+ \,\middle|\, \mathcal{F}_t \,\right]$$

and a put is denoted as

$$\mathcal{P}_t(T,K) := \mathbb{E}\left[\, (K - S_T)^+ \,\middle|\, \mathcal{F}_t \,\right] \ .$$

Recall that at any time $t > 0$, the quantity $V_t(t) = \int_0^t \zeta_s\,ds$ is the realized variance up to t. Hence, $V_t(T) - V_t(t)$ is the expectation of the remaining future value of a variance swap with maturity T.

A.1.1 Variance Swaps

PROPOSITION A.2 (Price and Hedge of a Variance Swap) *Let $K^* \in \mathbb{R}_{>0}$. The dynamic hedging strategy for a variance swap is given by*

$$\frac{1}{2}\int_t^T \zeta_s\,ds = F(S_T,K^*) - F(S_t,K^*) - \int_t^T \left(\frac{1}{K^*} - \frac{1}{S_u}\right) dS_u \tag{A.1}$$

with $F(x,K^*) := x/K^* - \log x/K^* - 1$.

The payoff $F(S_T,K^*) - F(S_t,K^*)$ can be replicated by a static portfolio of European options such that the price of a variance swap with maturity T at time t is given as

$$
\begin{aligned}
V_t(T) - V_t(t) \;\;=\;\; & 2\int_{K^*}^{\infty}\frac{1}{K^2}\mathcal{C}_t(T,K)\,dK + 2\int_0^{K^*}\frac{1}{K^2}\mathcal{P}_t(T,K)\,dK \\
& -2F(S_t,K^*) \ .
\end{aligned} \tag{A.2}
$$

Note the previous proposition also shows that both "cash-delta" (delta times the stock price) and "cash-gamma" (the derivative of delta with respect to S, multiplied by S^2) of a variance swap are constant.

Proof – By definition (1.1), a variance swap pays out the quadratic variation of $\log S$. Applying Itô to F as defined in the proposition yields (A.1).

Applying proposition A.1 to $F(\cdot, K^*)$ with $\partial_x F(x, K^*) = 1/K^* - 1/x$ and $\partial^2_{xx} F(x) = 1/x^2$ for $x_0 = K^*$ gives

$$F(S_T, K^*) - 0 = 0 + \int_0^{K^*} \frac{1}{K^2}(K - S_T)^+ \, dK + \int_{K^*}^{\infty} \frac{1}{K^2}(S_T - K)^+ \, dK$$

since $F(K^*, K^*) = \partial_x F(K^*, K^*) = 0$. Taking conditional expectation yields the result. $\square$

Apart from the mathematical pleasing result that variance swaps can be priced using European options, we want to stress that the hedging strategy for a variance swap suggested by proposition works *extremely* well in practise.[2] As an example, we have used all historic STOXX50E spot prices from January 1st 1992 to December 12th 2005 and have back-tested the discrete version of (A.1): to this end, we have computed the floating 90-day realized variance for $n = 1/1/1992$ to $n = 8/8/2005$,

$$\sum_{i=0}^{89} \left(\log \frac{S_{T_{n+i+1}}}{S_{T_{n+i}}} \right)^2 , \tag{A.3}$$

as well as its discrete hedge,

$$-2 \log \frac{S_{T_{n+90}}}{S_{T_n}} + 2 \sum_{i=0}^{89} \frac{1}{S_{T_{n+i}}} \left(S_{T_{n+i+1}} - S_{T_{n+i}} \right) . \tag{A.4}$$

(i.e. we have assumed that we purchased a log-contract up front). The impressive result is shown in figure A.1. The results are similar for all major indices.

Remark A.3 *In practical applications, it is more convenient to approximate* $\mathbb{E}[F(S_T)]$ *from above using a sequence of call and puts:*

If f *is a convex function with minimum* $f(x_0) = 0$ *in* x_0*, and if* $K^* = K_0^c < K_1^c < \cdots < K_m^c$ *and* $K^* = K_0^p > K_1^p > \cdots > K_m^p$*, then*

$$f(x) \le \sum_{i=1}^{m} w_i^c (x - K_i^c)^+ + \sum_{i=1}^{m} w_i^p (K_i^p - x)^+$$

for all $K_m^p \le x \le K_m^c$ *where we define* $w_0^c = w_0^p = 0$ *and then inductively*

$$w_i^c := \frac{f(K_i^c) - f(K_{i-1}^c)}{K_i^c - K_{i-1}^c} - \sum_{j=1}^{i-1} w_j^c$$

$$w_i^p := -\frac{f(K_i^p) - f(K_{i-1}^p)}{K_i^p - K_{i-1}^p} - \sum_{j=1}^{i-1} w_j^p .$$

For the variance swap with $F(x) := x - \log x - 1$, this approximation is shown in figure A.2.

[2] This is particularly interesting in the light of remark 7.26 on page 125.

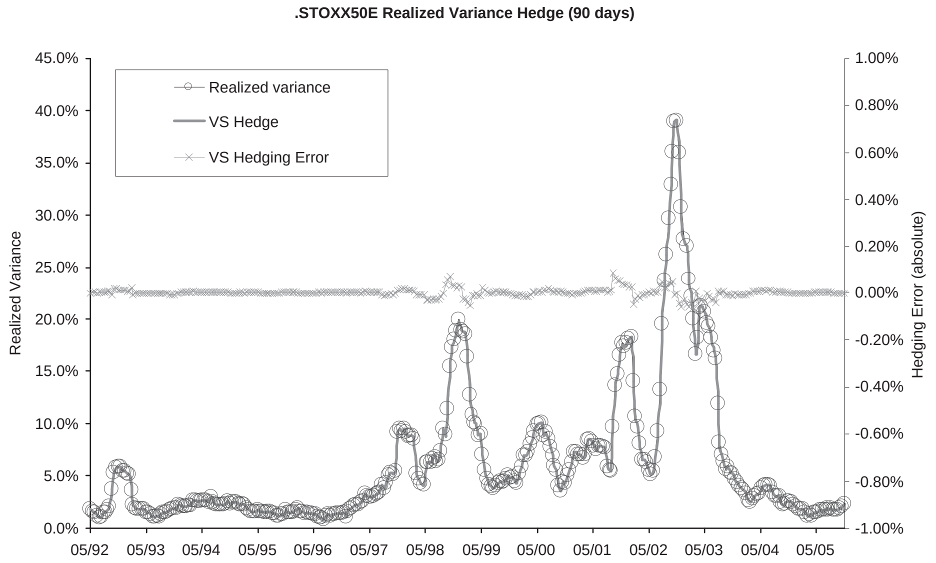


Figure A.1: Realized variance versus its hedge (A.4).

Remark A.4 *It should be noted that for continuous processes,*

$$\langle \log S\rangle_T = \int_0^T d\langle \log S\rangle_t = \int_0^T \frac{d\langle S\rangle_t}{S_t^2} \ .$$

Therefore, some variance swap contracts specify the payoff

$$\frac{d}{n}\sum_{i=1}^n \left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2$$

instead of (1.1) on page 10. These formulations are not equivalent in the presence of dividends.

Remark A.5 *The case of discrete cash dividends is discussed in depth in Bermudez et al. [BBFJLO06].*

Remark A.6 (Market Conventions) *In real markets, the transaction size for a variance swap is denoted in "vega" units. The heuristic idea is as follows: if σ denotes the volatility of the variance swap (i.e. the fair value is σ^2), then the variance swap has a "vega" of 2σ.*

Hence, if we were to protect a portfolio with an exposure of $\mathcal{V}$ "vega" via variance swaps, we would have to buy $\mathcal{N} = \frac{\mathcal{V}}{2\sigma}$ contracts. The quantity $\mathcal{N}$ is the actual notional of the trade, given the requested "vega".

A.1.2 Entropy Swaps

Recall from definition 5.14 on page 78 that the payoff of an *entropy swap* is defined as

$$\int_0^T S_t \,\langle \log S\rangle_t = \int_0^T S_t\zeta_t\, dt \ . \tag{A.5}$$

Proposition A.7 (Price and Hedge of an Entropy Swap) *Let $K^* \in \mathbb{R}_{>0}$ and define $G(x,K^*) := x\log(x/K^*) - x + K^*$. The dynamic hedging strategy for an entropy swap is given by*

$$\frac{1}{2}\int_t^T S_u\zeta_u\, du = G(S_T,K^*) - G(S_t,K^*) - \int_t^T \log\frac{S_u}{K^*}\, dS_u \ . \tag{A.6}$$

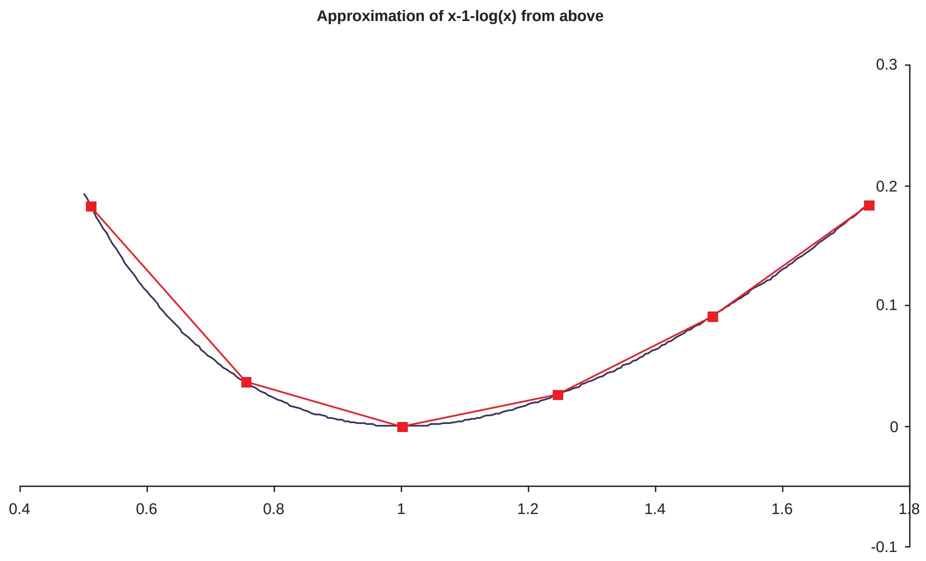


Figure A.2: Approximation of $F(x) := x - \log x - 1$ from above.

The payoff $G(S_T, K^*) - G(S_t, K^*)$ can be statically hedged with a portfolio of European options such that the price of the entropy swap is

$$\begin{aligned} U_t(T) - U_t(t) &= 2\int_{K^*}^{\infty} \frac{1}{K}\mathcal{C}_t(T,K)\,dK + 2\int_0^{K^*} \frac{1}{K}\mathcal{P}_t(T,K)\,dK \\ &\quad -2G(S_t, K^*)\ . \end{aligned} \tag{A.7}$$

Proof – Note that $\partial_x G(x, K^*) = \log x/K^*$ and $\partial^2_{xx} G(x, K^*) = 1/x$. Itô shows (A.6). Since $G(K^*) = \partial_x G(K^*) = 0$, proposition A.1 yields

$$G(S_T) - 0 = 0 + \int_0^{K^*} \frac{1}{K}(K - S_T)^+\,dK + \int_{K^*}^{\infty} \frac{1}{K}(S_T - K)^+\,dK$$

which proves the claim. □

As in the case of a variance swap, an entropy swap is usually priced using the approach discussed in remark A.3: figure A.3 shows the approximation of the function G. However, in practise an entropy swap barely trades because unlike the variance swap, its contract is not insensitive to certain dividend assumptions. This will be discussed in section A.2.

A.1.3 Shadow Options

It is instructive to consider an alternative proof of proposition A.7. To this end, note that at time t, we have

$$\begin{aligned} \mathcal{C}_t(T,K) &= \mathbb{E}\left[\,(S_T - K)^+ \,\middle|\, \mathcal{F}_t\,\right] \\ &= K S_t \mathbb{E}^S\left[\left(\frac{1}{K} - \frac{1}{S_T}\right)^+ \middle|\, \mathcal{F}_t\right] \\ &= K S_t\, \mathcal{P}^S_t\,(T, 1/K) \end{aligned}$$

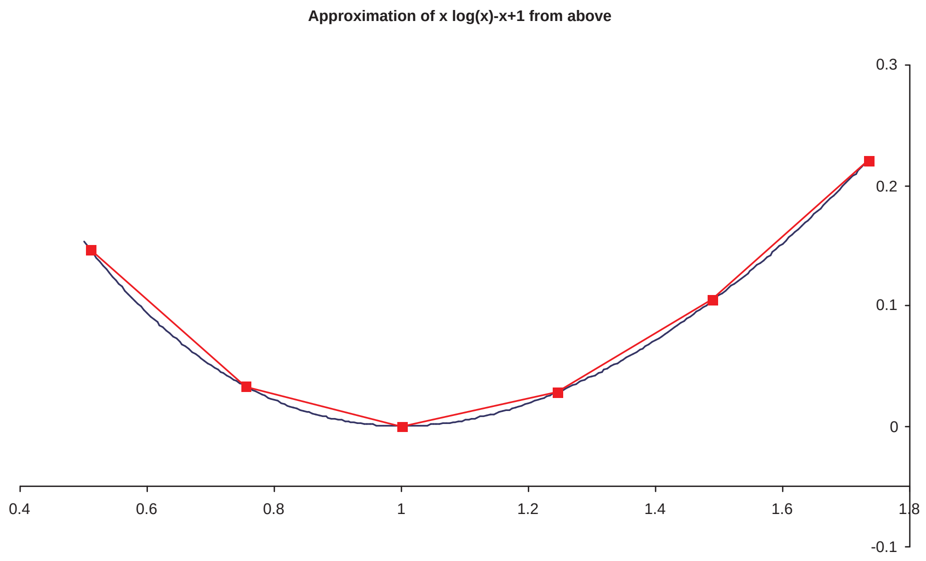


Figure A.3: Approximation of $G(x, K^*) := x\log(x/K^*) - x + K^*$ from above. Note the striking similarity to figure A.2.

where $\mathcal{P}_t^S(T,k)$ denotes the "shadow" put with maturity T and strike $1/K$, written on the $\mathbb{P}^S$-martingale S^{-1} (as before, the measure $\mathbb{P}^S$ is defined by $\mathbb{P}^S[A] := \mathbb{E}\left[\, 1_A\, S_t \,\right]$ for $A \in \mathcal{F}_t$).

We just have shown:

Lemma A.8 (Shadow Prices of Vanilla Options) *The* shadow prices *of vanilla options are given as*

$$\left.\begin{array}{rcl} \mathcal{C}_t^S(T,k) & = & \frac{k}{S_t}\mathcal{P}_t(T,1/k) \\ \mathcal{P}_t^S(T,k) & = & \frac{k}{S_t}\mathcal{C}_t(T,1/k) \end{array}\right\} \tag{A.8}$$

and are therefore available from the market.

Now,

$$\frac{U_t(T)}{S_t} = \mathbb{E}^S\left[\int_0^T \zeta_s\, ds \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}^S\left[\, \langle \log S^{-1}\rangle_T \,\middle|\, \mathcal{F}_t\,\right] \tag{A.9}$$

is just a variance swap on the martingale S^{-1} under $\mathbb{P}^S$.

Hence, we can use (A.2) under $\mathbb{P}^S$ and obtain for the strike $1/K^*$

$$\begin{aligned} \frac{U_t(T)}{S_t} - \frac{U_t(t)}{S_t} &= 2\int_0^{1/K^*} \frac{1}{k^2}\mathcal{C}_t^S(T,k)\, dk + 2\int_{1/K^*}^{\infty} \frac{1}{k^2}\mathcal{P}_t^S(T,k)\, dk \\ &\quad -2F(1/S_t, 1/K^*) \\ &= \frac{2}{S_t}\int_0^{1/K^*} \frac{1}{k}\mathcal{P}_t(T,1/k)\, dk + \frac{2}{S_t}\int_{1/K^*}^{\infty} \frac{1}{k}\mathcal{C}_t(T,1/k)\, dk \\ &\quad -2F(1/S_t, 1/K^*) \\ &= \frac{2}{S_t}\int_0^{1/K^*} k\frac{1}{k^2}\mathcal{P}_t(T,1/k)\, dk + \frac{2}{S_t}\int_{1/K^*}^{\infty} k\frac{1}{k^2}\mathcal{C}_t(T,1/k)\, dk \\ &\quad -2F(1/S_t, 1/K^*) \\ &\overset{(*)}{=} \frac{2}{S_t}\int_0^{K^*} \frac{1}{K}\mathcal{P}_t(T,K)\, dK + \frac{2}{S_t}\int_{K^*}^{\infty} \frac{1}{K}\mathcal{C}_t^S(T,K)\, dK \end{aligned}$$

$$-2F(1/S_t, 1/K^*)$$

In $(*)$ we have substituted $K := 1/k$, i.e. $dK = -1/k^2\,dk$. Multiplying by S_t yields the desired result (A.7) since

$$\begin{aligned} S_t F(1/S_t, 1/K^*) &= S_t\left(\frac{K^*}{S_t} - \log\frac{K^*}{S_t} - 1\right) \\ &= K^* + S_t \log\frac{S_t}{K^*} - S_t \;=\; G(S_t, K^*)\ . \end{aligned}$$

□

Let us also mention another interesting application of "shadow options", which can be formulate in more general contexts.

PROPOSITION A.9 (Delta in Stochastic Volatility Models) *Assume that S is a martingale and that the density of S_T/S_0 does not depend on S_0. In this case, the "Delta" of European options is readily available from the market as*

$$\partial_{S_0}\mathcal{C}_0(T,K) = \partial_k \mathcal{P}_0^S\left(T, \frac{1}{K}\right) = \frac{1}{S_0}\left(\mathcal{C}_0\,(T,K) - \frac{K}{S_0}\,(\partial_K \mathcal{P}_0)\,(T,K)\right)\ . \tag{A.10}$$

This proposition applies for example for a stroke Markov variance curve model (G, Z, ρ) (cf. definition 2.22) whose correlation functional ρ does not depend on S. It also applies to more general processes such as jump processes etc.

Proof – Let $X_t := S_T/S_0$. We have

$$\begin{aligned} \partial_{S_0}\mathcal{C}_0(T,K) &= \partial_{S_0}\mathbb{E}\left[\,(S_0 X_T - K)^+\,\right] \\ &= \partial_{S_0}\left(S_0\mathbb{E}\left[\left(X_T - \frac{K}{S_0}\right)^+\right]\right) \\ &= \frac{1}{S_0}\mathcal{C}_0\,(T,K) + \frac{K}{S_0^2}\mathbb{E}\left[\,1_{S_T\le K}\,\right] \\ &= \frac{1}{S_0}\mathcal{C}_0\,(T,K) - \frac{K}{S_0^2}\,(\partial_K \mathcal{P}_0)\,(T,K)\ . \end{aligned}$$

All quantities on the right hand side can be observed on the market. □

A.2 Deterministic Interest Rates and Proportional Dividends

It has been argued throughout the main text and in the previous section, that the assumption of deterministic interest rates, deterministic repo rates and deterministic proportional dividends is essentially equivalent to assuming no interest rates, repo rates or dividends when it comes to pricing European options or options on realized variance. We will show here how a market of European options and variance swaps on a dividend paying stock in a deterministic interest rate environment can be transformed into a market without dividends and no interest rates. The following discussion and a fundamental generalization to the case of deterministic cash dividends can be found in Bermudez/Buehler/Ferraris/Jordinson/Overhaus/Lamnouar [BBFJLO06].

We will also show that the situation is less clear-cut for *entropy swaps*. Indeed, we will introduce what we will call a *gamma swap*, which is a version of an entropy swap which accounts for dividends.

ASSUMPTIONS 7 *In this section, we assume that a deterministic short interest rate $r = (r_t)_{t\geq 0}$ prevails in the market. We also assume that holding the stock $S = (S_t)_{t\geq 0}$ earns the holder a proportional* repo rate $\mu = (\mu_t)_{t\geq 0}$ *(this rate might be negative if holding the stock inflicts costs). Finally, the stock is also assumed to pay discrete proportional dividends.*

We model the proportional dividends as follows: at each of the ex-dividend dates $0 = \tau_0 < \tau_1 < \cdots$ we assume that the stock price S jumps according to

$$S_{\tau_k} = S_{\tau_k -} e^{-D_k} \ .$$

where $S_{t-} := \lim_{s\uparrow t} S_t$.

In this situation, the stock price S is given in terms of its "martingale part" M and forward F

$$S_t = F_t M_t \ .$$

The forward is determined by standard no-arbitrage arguments (cf. Hull [H05]) as

$$F_t = \exp\left\{ \int_0^t (r_s - \mu_s)\, ds \right\} \mathcal{D}_t \quad \text{with} \quad \mathcal{D}_t := \exp\left\{ - \sum_{k:\tau_k \leq t} D_k \right\} .$$

We denote by $\mathrm{DF}_T := e^{-\int_0^T r_t\, dt}$ the discount factor from T.

The idea is now to reduce a market of European options and variance swaps on the underlying S to a market in terms of the driftless price process M.

ASSUMPTION 8 *We assume that the market of the stock and all observed European options is complete and that M is a true martingale under the unique pricing measure $\mathbb{P}$.*

Proposition 2.2 shows that then there exists a Brownian motion B and a short variance $\zeta \in L^{\mathrm{loc}}(B)$ such that

$$M_t = \mathcal{E}_t \left(\int_0^{\cdot} \sqrt{\zeta_s}\, dB_s \right) \ .$$

A.2.1 European Options

First, let us focus on European options. By put-call parity, it is clear that it is sufficient to discuss on European calls. We therefore assume that for all strikes $\mathbb{K}$ and all maturities T, European options $\mathbb{C}(T,\mathbb{K})$ on S are traded. The price of the call $\mathbb{C}(T,\mathbb{K})$ is given as

$$\mathbb{C}(T,\mathbb{K}) = \mathrm{DF}_T\, \mathbb{E}\left[(S_T - \mathbb{K})^+ \right] \ .$$

A simple transformation shows that a call on M with strike K and the same maturity T can be computed as

$$\mathcal{C}(T,K) := \mathbb{E}\left[(M_T - K)^+ \right] = \frac{1}{\mathrm{DF}_T F_T} \mathbb{C}\left(T, K F_T\right) \ . \tag{A.11}$$

Hence, a complete surface of European call prices on S yields also a complete surface of call prices on M.

A.2.2 Variance Swaps

Regarding variance swaps, we stick to the market convention that a typical variance swap will "take out" the dividends in the variance estimation. This is achieved by adding back the dividend into the return computation: in case of our stock S, a variance swap with maturity T and business days $0 = t_0 < \cdots < t_N = T$ pays therefore

$$\mathcal{V}^N(T) := \sum_{i=1}^{N} \left(\log \frac{S_{t_i-}}{S_{t_{i-1}}} \right)^2 . \tag{A.12}$$

As before, this quantity converges against

$$\mathcal{V}^N(T) \overset{N\uparrow\infty}{\longrightarrow} \int_0^T \zeta_s \, ds = \langle \log S^c \rangle_T$$

where S^c represents the continuous part of S, which is in our situation given as

$$S_t^c = \frac{S_t}{\mathcal{D}_t} = S_0 \exp\left\{ \int_0^t (r_s - \mu_s)\, ds \right\} \mathcal{E}_t \left(\int_0^{\cdot} \sqrt{\zeta_s}\, dB_s \right) .$$

Since S^c is continuous, we can apply the usual Itô-formula and obtain

$$\begin{aligned}
\langle \log S^c \rangle_T &= \int_0^T \frac{1}{(S_{t-}^c)^2}\, d\langle S^c \rangle_t \\
&= -2\log S_T^c + 2\log S_0^c + 2\int_0^T \frac{1}{S_{t-}^c}\, dS_t^c \\
&= -2\log S_T + 2\log \mathcal{D}_T + 2\log S_0^c + 2\int_0^T (r_t - \mu_t)\, dt + 2\int_0^T \sqrt{\zeta_t}\, dB_t \\
&= -2\log S_T + 2\log F_T + 2\int_0^T \sqrt{\zeta_t}\, dB_t .
\end{aligned}$$

Taking expectations yields the same result as in proposition A.2 (the computation of $\mathbb{E}[\log S_T]$ in terms of European options is independent of the actual dynamics of S as long $\log S_T$ is integrable at all).[3]

As a result, if European options with all strikes are traded at maturity T, we can price the variance swap again by pricing a log-contract.

A.2.3 Gamma Swaps

The case of an *entropy swap* is less clear-cut mainly because of the way the contract can be formulated. Let us define the payoff of an entropy swap just by adjusting the forward:

$$\mathcal{U}^N(T) := \sum_{i=1}^{N} \frac{S_{t_i}}{F_{t_i}} \left(\log \frac{S_{t_i}/F_{t_i}}{S_{t_{i-1}}/F_{t_{i-1}}} \right)^2 = \sum_{i=1}^{N} M_{t_i} \left(\log \frac{M_{t_i}}{M_{t_{i-1}}} \right)^2 . \tag{A.13}$$

This simply recovers the original contract on the driftless underlying M. Since we have shown in section A.2.1 that we can also recover European option prices on M, we are able to compute the price $U_0(T) := \mathrm{DF}_T\, \mathbb{E}^S \left[\int_0^T \zeta_t \, dt \right]$ of an entropy swap using proposition A.7.

[3] Note that if the dividends D_k are stochastic, then $\mathbb{E}[\log F_T] \neq \log \mathbb{E}[F_T]$.

However, from an investor's perspective, (A.13) is quite an unsatisfying formulation. The martingale part M cannot be observed directly in the market, so the payoff looks very artificial. This is alleviated by using a *gamma swap* or *weighted variance swap* (see appendix B.2) with payoff

$$\mathcal{G}^N(T) := \sum_{i=1}^{N} \frac{S_{t_i}}{S_0} \left(\log \frac{S_{t_i-}}{S_{t_{i-1}}} \right)^2 . \tag{A.14}$$

The name "gamma swap" comes from the observation that it has nearly a linear cash-gamma (actually, an entropy swap has a linear cash-gamma, but as we mentioned, such a product is not attractive to investors). In the limit, we have

$$\mathcal{G}^N(T) \stackrel{N\uparrow\infty}{\longrightarrow} \int_0^T \frac{S_t}{S_0} d\langle \log S^c \rangle_t .$$

Since

$$\int_0^T \frac{S_t}{S_0}\, d\langle \log S^c \rangle_t = \int_0^T \frac{S_t}{S_0} \zeta_t \, dt ,$$

we get

$$\mathbb{E}\left[\int_0^T \frac{S_t}{S_0} d\langle \log S^c \rangle_t \right] = \int_0^T \mathbb{E}^S \left[\frac{F_t}{S_0} \zeta_t \right] dt = \int_0^T \frac{F_t}{S_0}\, \mathbb{E}^S \left[\zeta_t \right] dt .$$

The price of a gamma swap is therefore given by

$$\Gamma_0(T) := \mathrm{DF}_T\, \mathbb{E}\left[\int_0^T \frac{S_t}{S_0}\, \zeta_t \, dt \right] = \mathrm{DF}_T \int_0^T \frac{F_t}{S_0}\, \partial_T \frac{U_0(T)}{\mathrm{DF}_T} \bigg|_{T=t} dt \tag{A.15}$$

where $U_0(T)$ denotes as before the price of an entropy swap.

REMARK A.10 *In practise, (A.15) is computed using a time-discretization so that the gamma swap is a series of weighted entropy swaps.*

Regarding hedging, the situation is less complicated. Once the contract is evaluated, Itô's lemma shows that

$$\frac{1}{2} \int_0^T \frac{S_t}{S_0} d\langle S_t \rangle = - \int_0^T \log S_t \, dS_t + \Big\{ (S_T \log S_T - S_T) - (S_0 \log S_0 - S_0) \Big\} . \tag{A.16}$$

As for the variance swap, we can back-test this strategy using its discrete version (cf. figure A.1 above). Under the assumption that we were able to buy the contract $(S_T \log S_T - S_T)$ and that all our delta-hedging is covered by the initial premium, we again find that this hedge works very well, as figure A.4 shows.

Moreover, figure A.5 shows, finally, the historical returns from both variance swaps and gamma swaps versus the returns from the stock. In the current low volatility environment, both payoffs behave very similar.

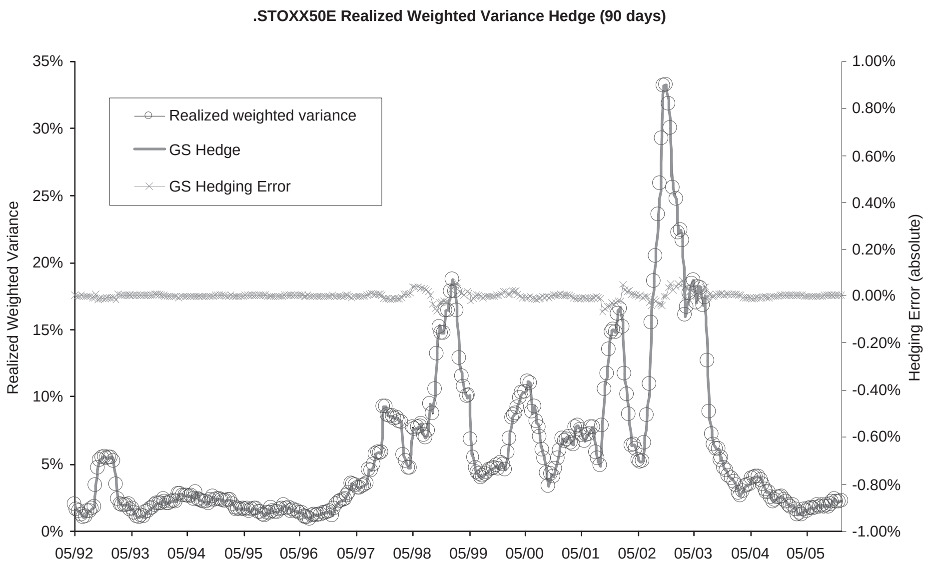


Figure A.4: Realized weighted variance versus its hedge (A.16).

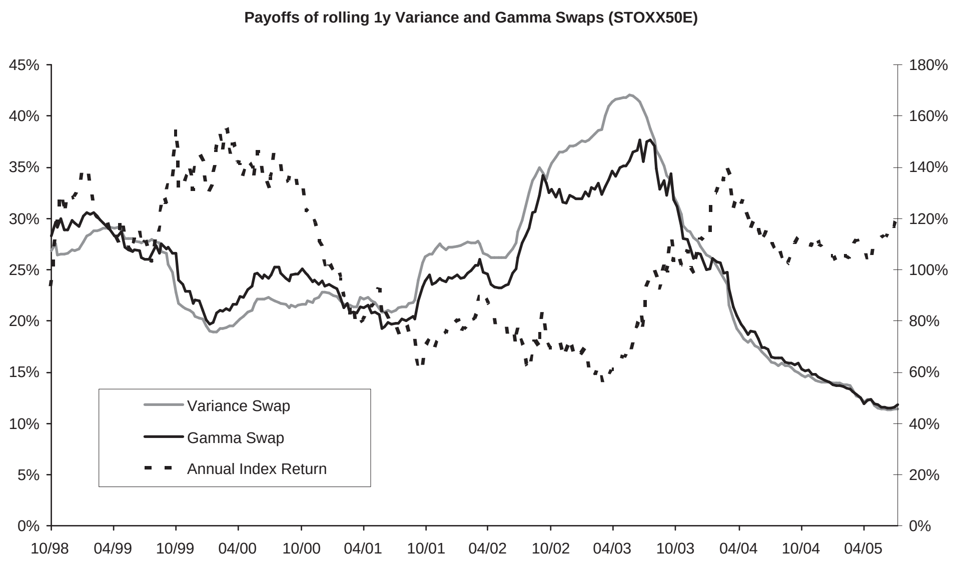


Figure A.5: Historic payoffs of variance and gamma swaps for STOXX50E since January 1992. Both are mostly anti-correlated with index returns.

Appendix B

Example Term Sheets

In this chapter we provide a few real-life term sheets for common volatility products. The term sheets are product sheets from Deutsche Bank's Global Equity Derivatives (GED) Equity Structuring Group. This group is now under the same umbrella as the former GED Global Quantitative Research Team, and referred to as the Quantitative Products team.

These documents have been prepared for information purposes only. They do not constitute an offer, solicitation, or an invitation to make an offer, to buy or sell any security or financial instrument in any jurisdiction.

EQUITY STRUCTURING GROUP : VARIANCE SWAP

EQUITY STRUCTURINGgroup

GED

Variance Swap

The Variance Swap allows clients to invest directly in the realized variance of an underlying index, stock or basket. The advantage this product has over option strategies is that no other optionality risks are embedded. The payout (to receive) or the payment (to make) at maturity is the difference between the realized volatility and the agreed initial reference level. (To learn more about the difference between a variance and a volatility swap please consult the "Technical Insight" section of our website).

Key Characteristics

- Allows a view to be taken on realized variance
- This structure is a swap and as such is not principle protected

Pricing Reference

• Instrument	Swap
• Maturity	1 Year
• Currency	EUR
• Number of Underlyings	1
• Underlyings	Nokia (NOK1v.FH)
• Vega notional	N EUR (i.e. N * 1 volatility point)

Formula

At maturity the long swap counterparty receives (or pays if the amount is negative) the following amount in EUR:

$$\text{Notional} \times \left(\text{Min}\left(\text{Vol}_{\text{Realised}}^{2},\ (2.5 \times \text{Vol}_{\text{Set}}^{2}\right) - \text{Vol}_{\text{Set}}^{2} \right)$$

Where:

Vol set	Is the predefined strike level
Vol realized	$\sqrt{252} \times \sqrt{\dfrac{\sum_{i=1}^{i=Days} \left(Return(t_i) - Return_{mean}\right)^2}{Days - 1}}$
$\text{Return}(t_i)$	$\ln\left(\dfrac{Underlying(t_i)}{Underlying(t_{i-1})}\right)$
$\text{Return}_{\text{mean}}$	Zero Mean
$\text{Underlying}(t_i)$	Official Closing Price of the Underlying on the Averaging Date t_i
$\text{Underlying}(t_0)$	Official Closing Price of the Underlying on the Start Date t_0
$t_i\,(i = 1 \ldots Days)$	Averaging Dates, every exchange business day between the Start Date (excluded) to the Final Valuation Date (included).
Days	Number of Averaging Dates

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Figure B.1: Term sheet for a standard variance swap.

Global Equity Derivatives

Gamma Swap linked to .STOXX50E Index

Draft Terms 25 July 2005 –
(Zero Mean / Daily Observations)

The Price Weighted Variance Swap has similar characteristics to a standard Variance Swap. The Buyer receives the realized daily variance of the Reference Security weighted by the previous day's closing price, and thus has exposure to its price path. Price Weighted Variance Swaps often cost less than standard Variance Swaps when the Reference Index has a downward-sloping skew of implied volatility.

Instrument Type	Index Price-Weighted Variance Swap ("Gamma Swap")
Party A (Swap Seller)	Deutsche Bank AG London Branch
Party B (Swap Buyer)	XYZ Client
Currency	EUR
Vega Notional	EUR XXX for 1 volatility point
Notional	EUR XXX which is equivalent to $\frac{100 \; Vega}{2 \; Vol_{Set}}$
Trade Date	25 July 2005
Start Date	25 July 2005
Final Valuation Date	16 June 2006
Maturity Date	3 Currency Business Days after the Final Valuation Date
Underlying	Euro Stoxx 50 Index (Bloomberg code: SX5E <Index>)
Vol_{Set}	**ZZ%**
Equity Amount	A Euro amount equal to Notional $(Vol_{Realised}^2 - Vol_{Set}^2)$ Where : $Vol_{Realised} = \sqrt{252}\sqrt{\frac{\sum_{i=1}^{Days}\left(\frac{Underlying\ t_i}{Underlying\ t_0}\ Return\ t_i\right)^2}{Days}}$ $Return\ t_i = \ln\frac{Underlying\ t_i}{Underlying\ t_{i-1}}$ $Return_{mean}$ = Zero Mean $Underlying\ t_i$ = Closing of the Underlying on the Averaging Date t_i $Underlying\ t_0$ = Closing of the Underlying on the Start Date t_0 $t_i\ (i = 1 \ldots Days)$ = Averaging Dates, every exchange business day between the Start Date (excluded) to the Final Valuation Date (included). Days = Number of Averaging Dates

Figure B.2: The term sheet for a gamma swap.

EQUITY STRUCTURING GROUP : VARIANCE SWAP OPTION

EQUITY STRUCTURINGgroup

GED

Variance Swaption

This product has all of the attributes of a variance swap plus the additional benefits of an option, as opposed to a swap structure (where the investor may show losses at maturity). Again the variance swap allows clients to invest directly in the realized variance of an underlying index, stock or basket. The advantage this product has over option strategies is that no other optionality risks are embedded. The payout (to receive) or the payment (to make) at maturity is the difference between the realized volatility and the agreed initial reference level. (To learn more about the difference between a variance and a volatility swap please consult the "Technical Insight" section of our website).

Key Characteristics

- Allows a view to be taken on realized variance
- Final payoff is $\mathrm{MAX}(0, \mathrm{VOL}_{\mathrm{REALIZED}}^2 - \mathrm{VOL}_{\mathrm{STRIKE}}^2)$ – Premium Amount
- In this structure the investor can not loose more than the fixed premium paid for the option

Pricing Reference

Instrument	Option
Maturity	1 Year
Currency	EUR
Number of Underlyings	1
Underlyings	Nokia (NOK1v.FH)
Vega notional	N EUR (i.e. N * 1 volatility point)

Formula

At maturity the long swap counterparty receives (or pays if the amount is negative) the following amount in EUR:

$$\mathrm{MAX}(0, \mathrm{VOL}_{\mathrm{REALIZED}}^2 - \mathrm{VOL}_{\mathrm{STRIKE}}^2) - \text{Premium Amount}$$

Where:

Vol strike — Is the predefined strike level

Vol realized — $$\sqrt{\frac{252 \times \sum_{t=1}^{t=N} \ln\left(\frac{P_t}{P_{t-1}}\right)^2}{\text{Expected } N}}$$

Expected N — Is the expected number of Exchange business days from, but excluding the effective date, to and including, the valuation date

N — Is Expected N minus the total number of disrupted days between the effective date and the valuation date (if any).

Ln — Is the natural logarithm

Deutsche Bank

Figure B.3: Term sheet for a call on realized variance.

EQUITY STRUCTURING GROUP : VOLATILITY SWAP

EQUITY STRUCTURINGgroup

GED

Volatility Swap

This payoff allows investors to directly take a view on the realized volatility of an underlying index, stock or basket. The advantage that this product has over option strategies is that no delta rehedging etc is required. The payout (to receive) or the payment (to make) at maturity is the difference between the realized volatility and the agreed initial reference level.

Key Characteristics

- The settlement amount at maturity is the difference between ($VOL_{Realised}$ - VOL_{Set}) x 100 x Notional

Pricing Reference

• Maturity	1 year
• Currency	EUR
• Number of Underlyings	1
• Underlyings	.STOXX50E
• Observations	Daily

Formula

At maturity the following amount shall be settled:

$$\text{Notional Amount} * (VOL_{REALIZED} - VOL_{SET})$$

Where:

$VOL_{REALIZED}$	$\sqrt{\dfrac{252 \times \sum_{t=1}^{t=Days} \ln\left(\dfrac{\text{Relevant Price}_t}{\text{Relevant Price}_{t-1}}\right)^2}{\text{Days}}}$
VOL_{SET}	To be agreed, eg 20%
Days	*Days* is the number of Exchange Business Days from, but *excluding*, the Trade Date to, and *including*, the Valuation Date
Relevant Price$_0$	<price of Underlying at the time of the trade>
Relevant Price$_1$	The official closing level of the Underlying Index on the Exchange Business Day immediately following the Trade Date
Relevant Price$_t$	<relevant official closing or futures settlement level>
	(Relevant Prices are subject to market disruption language as set forth in the confirmation.)

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Figure B.4: Term sheet for volatility swap.

EQUITY STRUCTURING : NAPOLEON

EQUITY STRUCTURINGgroup

GED

Napoleon

This note gives an introduction to structures based on the Napoleon payoff. The Napoleon offers investors a potentially high coupon with exposure to market growth. This makes the product especially attractive to yield seeking investors expecting steady increases in the market and low volatility.

Key Characteristics

- Benefit from a range bound or increasing market when volatility is low
- Potentially high yield
- Cliquet feature means renewed chances every year

Pricing Reference

Maturity	5 years
Currency	EUR
Number of underlyings	1
Underlying	.STOXX50E
Observations	Monthly
Swap PV	14.68 (EUR Libor Flat - 5Y swap at 3.22%)
Note Book Open	Based on EUR Flat funding

Napoleon without Guaranteed Coupon

Each year the worst monthly performance is added to the 9% base coupon to form the annual coupon.

The payoff is defined by

Coupon(t) $$Max(0{,}9\% + Min_{i=1}^{12}(\frac{S_i}{S_{i-1}} - 1))$$

Note Book Open **97.91%**

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Figure B.5: Term sheet for a Napoleon.

Appendix C

Auxiliary Results

C.1 The Impact of incorrect Stochastic Volatility Dynamics

In this section, we want to present a standard "folklore" result which describes the impact of specifying a stochastic volatility wrongly. It shows the particular shape of the *profit/loss process* Γ which has been mentioned in section 5.1.

Assume that the real-market stock price process and its short-variance on $\mathbb{W}$ are given as the unique solution to the SDE

$$\begin{aligned} d\zeta_t &= a_t(\zeta_t)\,dt + b_t(\zeta_t)\,dW_t^1 \\ dS_t &= S_t\sqrt{\zeta_t}\,d\left(\rho_t W_t^1 + \sqrt{1-\rho_t^2}W_t^2\right) \ . \end{aligned}$$

Also assume that we are given a model which defines a stock price process $\bar{S}$ and a short-variance $\bar{\zeta}$ on a different stochastic base $\bar{\mathbb{W}}$ as the solution to the SDE

$$\begin{aligned} d\bar{\zeta}_\tau &= \bar{a}_\tau(\bar{\zeta}_\tau)\,d\tau + \bar{b}_t(\bar{\zeta}_\tau)\,d\bar{W}_\tau^1 \\ d\bar{S}_\tau &= \bar{S}_\tau\sqrt{\bar{\zeta}_\tau}\,d\left(\bar{\rho}_\tau \bar{W}_\tau^1 + \sqrt{1-\bar{\rho}_\tau^2}\bar{W}_\tau^2\right) \ . \end{aligned}$$

We assume that S and $\bar{S}$ are true martingales on their respective stochastic bases. For simplicity, we also assume that ρ and $\bar{\rho}$ are deterministic. For a bounded European payoff $\mathbb{H} : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ define

$$\bar{H}(\tau, s, z) := \bar{\mathbb{E}}\left[\, \mathbb{H}(\bar{S}_T) \,\middle|\, \bar{S}_\tau = s, \bar{\zeta}_\tau = z \,\right] \ .$$

This function solves the PDE

$$\begin{aligned} 0 \;=\;& \partial_\tau \bar{H}(\tau,s,z) + \partial_z \bar{H}(\tau,s,z)\,\bar{a}_\tau(z) \\ &+\frac{1}{2}\partial_{ss}^2 \bar{H}(\tau,s,z)\,s^2 z + \frac{1}{2}\partial_{zz}^2 \bar{H}(\tau,s,z)\,\bar{b}_\tau(z)^2 + \partial_{sz}^2 \bar{H}(\tau,s,z)\,\bar{\rho}_t\sqrt{z}\bar{b}_\tau(z) \end{aligned} \tag{C.1}$$

with boundary condition $\bar{H}(T,s,z) = \mathbb{H}(s)$.

Now assume that we mark the payoff constantly with the model, i.e. at all times t we assume the value of $\mathbb{H}$ is given as

$$\bar{H}\Big(t, S_t, \zeta_t\Big)$$

where S_t and ζ_t are the market quantities (we can deduce ζ_t from the known past path of S). Via Itô,

$$d\bar{H}\Big(t, S_t, \zeta_t\Big) \;= dM_t \quad +\partial_\tau \bar{H}(t, S_t, \zeta_t)\,dt$$

$$
\begin{aligned}
&+\partial_z \bar{H}(t, S_t, \zeta_t)\, a_t(\zeta_t)\, dt \\
&+\frac{1}{2}\partial^2_{ss}\bar{H}(t, S_t, \zeta_t)\, S_t^2 \zeta_t\, dt \\
&+\frac{1}{2}\partial^2_{zz}\bar{H}(t, S_t, \zeta_t)\, b_t(\zeta_t)^2\, dt \\
&+\partial^2_{sz}\bar{H}(t, S_t, \zeta_t)\, \rho_t \sqrt{\zeta_t} b_t(\zeta_t)\, dt \ .
\end{aligned}
$$

where M is a local martingale. Now we use (C.1) and replace $\partial_\tau \bar{H}(t, S_t, \zeta_t)$ such that we obtain

$$
\begin{aligned}
d\bar{H}\Big(t, S_t, \zeta_t\Big) \quad = dM_t \quad &+\partial_z \bar{H}(t, S_t, \zeta_t)\ \left(a_t(\zeta_t) - \bar{a}_t(\zeta_t)\right)\ dt \\
&+\frac{1}{2}\partial^2_{zz}\bar{H}(t, S_t, \zeta_t)\ \left(b_t(\zeta_t)^2 - \bar{b}_t(\zeta_t)^2\right)\ dt \\
&+\partial^2_{sz}\bar{H}(t, S_t, \zeta_t)\ \sqrt{\zeta_t}\ \left(\rho_t b_t(\zeta_t) - \bar{\rho}_t \bar{b}_t(\zeta_t)\right)\ dt \ .
\end{aligned}
$$

This computation shows the form of the "profit/loss" process in this simple example (it is the sum of the terms of finite variation above). It also shows that it is particularly important to capture the market variance properly in a region where the derivatives of $\bar{H}$ with respect to ∂_z, ∂^2_{zz} and ∂^2_{zs} are large.

Appendix D

Transition Densities under Constraints

In this appendix, we will pick up the construction of *relative upper pricing measures* $\bar{\mu} = (\bar{\mu}^\tau)_{\tau \in \mathcal{T}}$ for a discrete set of European call prices. It has been shown in section 7.3, page 107ff. how a set $\mathcal{C} = (\mathcal{C}_\ell)_{\ell=1,\dots,d_C}$ of call prices can be checked for strict absence of arbitrage. In this case, the measures $\bar{\mu}^\tau$ are *compatible* with the observed call prices in the sense that

$$\mathcal{C}(\tau, K) = \int (x - K)\, \bar{\mu}^\tau(dx)$$

for all $K \in \mathcal{K}^\tau$ (we stick to the notation of section 7.3 on page 106ff). We have also discussed how a set $\mathcal{C}$ which is not strictly arbitrage-free can be modified to find a "close" strictly arbitrage-free call price surface.

In this appendix, we will now demonstrate how we construct transition matrices Π^j such that

$$\mu^j = \Pi^{j\prime} \mu^{j-1}$$

for $j = 1, \dots, d_\tau$ for a given strictly arbitrage-free set of measures $\mu = (\mu^\tau)_{\tau \in \mathcal{T}}$. This in turn defines a discrete-state discrete-time Markov martingale S with these marginal distributions μ^τ.

The following exposition closely follows [B06a], where questions of pricing and possible extensions are also discussed.

D.1 Expensive Martingales

Let us assume that we are given a sequence of strictly arbitrage-free measures $\mu = (\mu^\tau)_{\tau \in \mathcal{T}}$ (think of the relative upper pricing measures) with masses only in the strikes $0 =: K_0^\tau < \cdots < K_{d_\tau+1}^\tau := K^*$ for $\tau \in \mathcal{T}$ (recall that d_τ denotes the number of market strikes $\mathcal{K}^\tau$ and that K^* is the *zero price strike*).

We can now apply theorem 7.8 which asserts there must be a Markov process $S = (S_\tau)_{\tau \in \mathcal{T}}$ which reprices the market $\mathcal{C}$ implied by μ. As before, we use the indices $j = 1, \dots, d$ with $d := d_\tau := \#\mathcal{T}$ to refer to quantities related to the maturities $\tau^1, \dots, \tau^d$.

Let us denote the unit vector from $\mathbb{R}^{d_j+2}$ by 1^j for $j = 1, \dots, d$. We also use the notation $\mu^j = (\mu_0^j, \dots, \mu_{d_j+1}^j)'$ for the $d_j + 2$-dimensional column vector of point masses $\mu_i^j := \mu^j[K_i^j]$.

The transition-probabilities of S "from u to ℓ" under a measure $\mathbb{P}$ are

$$\Pi^j_{u\ell} := \begin{cases} \mathbb{P}[\ S_j = K^j_\ell \mid S_{j-1} = K^{j-1}_u\ ] & \text{if } \mathbb{P}[S_{j-1} = K^{j-1}_u] > 0, \\ 0 & \text{otherwise.} \end{cases}$$

for $j = 1, \dots, m$ (with, as usual, $S_j := S_{\tau_j}$ and $S_0 := 1$). This yields a matrix

$$\Pi^j := \begin{Bmatrix} \Pi^j_{0,0} & \cdots & \Pi^j_{0,d_j+1} \\ \vdots & & \vdots \\ \Pi^j_{d_{j-1}+1,0} & \cdots & \Pi^j_{d_{j-1}+1,d_j+1} \end{Bmatrix}$$

Hence a row represents the probabilities $\mathbb{P}[\ S_j \in dx \mid S_{j-1} = K^{j-1}_u\ ]$. We know that such a kernel exists, but how can we construct one? Let us formalize the notion of a stochastic kernel.

Definition D.1 *We call a Matrix* $\Pi^j = (\Pi^j_{u\ell})$ *with* $d_{j-1} + 2$ *rows and* $d_j + 2$ *columns a* Martingale-kernel *at* $\tau_j \in \mathcal{T}$ *iff*

(a) it is positive $\Pi^j_{u\ell} \geq 0$*,*

(b) it is a conditional probability $\Pi^j 1^j = 1^{j-1}$ *(all rows sum up to one) and*

(c) it has the martingale property $\Pi^j K^j = K^{j-1}$ *(the mean of row u is* K^{j-1}_u*).*

We call such a kernel compatible with μ *iff additionally*

(d) is a transition kernel for μ*, i.e.* $\Pi^{j\prime}\mu^{j-1} = \mu^j$*.*

(The initial kernel Π^1 *is just the transpose of* μ^1*.)*

Remark D.2 *The set of compatible Martingale-kernels* $\mathcal{P}$ *is a convex set.*

Definition D.3 (Most expensive martingales) *Given Martingale-kernels* $\Pi = (\Pi^j)_{j=1,\dots,m}$*, we call the Markov martingale S with* $S_0 := 1$ *and transition probabilities*

$$\mathbb{P}[\ S_j = K^j_\ell \mid S_{j-1} = K^{j-1}_u\ ] := \Pi^j_{u\ell}$$

the martingale of Π*. If Π is compatible with the relative upper pricing measures $\bar{\mu}$ of a market $\mathcal{C}$, then S is a* most expensive *martingale.*

D.1.1 Construction of a Transition Kernel

Now note that the properties of definition D.1 are in fact all linear conditions on each matrix Π^j. Indeed, let us fix some j (the notion of which we will omit in this subsection) and consider the column vector of rows of Π^i,

$$\kappa := \left(\Pi_{0,0}, \dots, \Pi_{0,d_j+1}; \Pi_{1,0}, \dots, \Pi_{1,d_j+1}; \Pi_{d_{j-1}+1,0}, \dots, \Pi_{d_{j-1}+1,d_j+1}\right)' \tag{D.1}$$

We have $\kappa \in \mathbb{R}^N$ with $N := (d_j + 2)(d_{j-1} + 2)$. The conditions *2* to *4* of definition D.1 can be written as

$$\begin{aligned} A\,\kappa &= x \\ B\,\kappa &= y\ . \end{aligned}$$

Here we use the $2(d_{j-1}+2) \times N$-matrix

$$A := \begin{pmatrix} 1^{j\prime} & 0^{j\prime} & & \dots & & & 0^{j\prime} \\ K^{j\prime} & 0^{j\prime} & & \dots & & & 0^{j\prime} \\ 0^{j\prime} & 1^{j\prime} & 0^{j\prime} & \dots & & & \vdots \\ 0^{j\prime} & K^{j\prime} & 0^{j\prime} & \dots & & & \vdots \\ \vdots & & & \dots & 0^{j\prime} & 1^{j\prime} & 0^{j\prime} \\ \vdots & & & \dots & 0^{j\prime} & K^{j\prime} & 0^{j\prime} \\ 0^{j\prime} & & & \dots & & 0^{j\prime} & 1^{j\prime} \\ 0^{j\prime} & & & \dots & & 0^{j\prime} & K^{j\prime} \end{pmatrix} \qquad x := \begin{pmatrix} 1 \\ K_0^{j-1} \\ 1 \\ K_1^{j-1} \\ \vdots \\ \vdots \\ 1 \\ K_{d_{j-1}+1}^{j-1} \end{pmatrix}$$

and the $(d_j+2) \times N$ matrix

$$B := \begin{pmatrix} v_0^{j-1} & 0 & 0 & \dots & 0 & v_1^{j-1} & 0 & \dots & & & 0 \\ 0 & v_0^{j-1} & 0 & \dots & 0 & 0 & v_1^{j-1} & 0 & \dots & & 0 \\ & & \ddots & & & & & \ddots & & & \\ & & & \ddots & & & & & & & \\ 0 & \dots & & 0 & v_0^{j-1} & 0 & \dots & & & 0 & v_{d_{j-1}+1}^{j-1} \end{pmatrix},$$

as well as

$$y := \begin{pmatrix} \mu_0^j \\ \vdots \\ \mu_{d_j+1}^j \end{pmatrix}.$$

Hence, define the $[(d_j+2)+2(d_{j-1}+2)] \times N$-matrix

$$M_1 := \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{and} \quad z_1 := \begin{pmatrix} x \\ y \end{pmatrix} \tag{D.2}$$

Now note that while the conditions encoded in $M_1\kappa = x_1$ admit at least one positive solution, they are not linearly independent. This is due to the fact that both μ^j and μ^{j-1} are probability measures and that both have unit expectation:

Since they are probability measures, we have

$$1^{j\prime}\mu^j = 1^{j-1\prime}\mu^{-1} = 1 \tag{D.3}$$

Now μ^j is given as

$$\mu^j = y = B\kappa$$

hence, say, $v_{d_{j-1}+1}^{j-1}$ can be expressed as a linear combination of v_u^{j-1} for $u = 0, \dots, d_{j-1}+1$.

The unit expectation of μ^j and μ^{j-1} on the other hand means

$$K^{j\prime}\mu^j = K^{j-1\prime}\mu^{-1} = 1$$

so we can express for example $v_{d_{j-1}}^{j-1}$ in terms of the other variables.

Consequently, we can reduce the system (D.2) to

$$M\kappa = z$$

by removing the last two rows of $(M_1|z_1)$.

This yields

CONCLUSION D.4 *To find martingale kernels* Π *which are compatible with* μ*, we have to solve the linear programming "feasibility" problems*

$$\begin{cases} M^j \kappa = z^j \\ \quad \kappa \geq 0 \end{cases} \tag{D.4}$$

for $M^j \in \mathbb{R}^{D^j \times N^j}$ *with* $D^j := (d_j + 2) + 2(d_{j-1} + 2) - 2$ *and* $N^j := (d_j + 2)(d_{j-1} + 2)$ *as above.*

This result is quite promising since linear programming problems can be solved efficiently and are well-studied. Given that the matrices M^j are very sparse, the solution of the LP problem above is usually solvable in reasonable time.

REMARK D.5 *For practical implementation, the matrices* M^j *can be further reduced by exploiting the following facts*

(a) For all states K_ℓ^j *with* $\mu_\ell^j = 0$ *(i.e. states which have no mass in* τ_j*), the column* $(K_{u\ell}^j)_{u=0,\dots,d^{j-1}+1}$ *can be ignored.*

(b) Equally, if $\mu_u^{j-1} = 0$*, then the entire uth row can be omitted.*

(c) The states 0 *and* K^* *are absorbing and the respective rows are therefore trivial.*

It also possible to limit the range of the conditional probabilities by imposing additional conditions. However, it is not clear to us yet how this can be achieved while ensuring that a solution to the new problem still exists.

D.2 Incorporating Weak Information

The previous section has shown how we can construct a "most expensive" finite-state martingale if we are given a strictly arbitrage-free market. However, the mere fact that usually $D^j \ll N^j$ means that the system (D.4) has many solutions. Indeed, remark D.2 shows that the set of solutions will be convex, hence as soon as there are just two possible solutions to (D.4), there will immediately be an infinite number of additional possibilities.

Also observe that most algorithms which solve linear-programming problems (see, for example Fang et al. [FP93]) will usually find extremal solutions.

Now, the various kernels Π which satisfy (D.4) differ in the way they evaluate non-European functionals (while they agree for all European options). We can therefore choose to impose further constraints to identify a particular kernel of interest.

REMARK D.6 *Note that the problem here is similar to a classical problem of pricing in an incomplete market, but under constraints. In fact, we try to single out one pricing measure out of a set of martingale measures.*

The difference is that the various measures here do not need to be equivalent to each other.

D.2.1 Mean-Variance Pricing

One way to identify a unique solution to (D.4) is to impose an additional optimality criterion. In principle, this could be some conditional mean-variance criterion.

The martingale property yields $\Pi^j K^j = K^i_{j-1}$ for each conditional expectation. The conditional variance of the martingale of Π is therefore

$$\varsigma_i^j := \mathbb{E}\left[\, S_j^2 - \mathbb{E}\left[\, S_j \mid S_{j-1} = K_i^{j-1} \,\right]^2 \,\middle|\, S_{j-1} = K_i^{j-1} \,\right] \ .$$

We can write this as

$$\varsigma_i^j = \Pi^j \, K^j I^j K^j - \left(K^i_{j-1}\right)^2$$

where I denotes the $d^j \times d^j$ unity matrix. This is a linear equation in Π. Hence, it is possible to minimize the variance over problem (D.4).

Other possibilities are possible. We want, however, concentrate on what we term "weak information".

D.2.2 Using Forward Started Call Prices

Let us fix some maturity τ_j. Assume that for this maturity, we have some "weak information": Approximate prices of options on S_j and S_{j-1}. For example prices which are *probably* correct or for which we have a good estimate (for example, over-the-counter products which are not liquidly traded and have high spreads). Let $\mathcal{F}^j = \{ \, f_i^j \ ; \ i = 1, \ldots, z_j \, \}$ be some functions

$$f_i^j \left(x_j, x_{j-1}\right) \ .$$

For example, these could be some "forward start calls"

$$f_i^j \left(x_j, x_{j-1}\right) := \left(\frac{x_j}{x_{j-1}} - h_i\right)^+ 1_{x_{j-1}>0}$$

with strikes $h \in \{h_1, \ldots, h_{z_j}\}$. The price of such a function f under given pricing kernels $\Pi = (\Pi^j)_{j=1,\ldots,d}$ compatible with some measures $\mu = (\mu^j)_{j=1,\ldots,d}$ is then given as

$$\sum_{u=0}^{d_{j-1}+1} \mu_u^{j-1} \sum_{\ell=0}^{d_j+1} \Pi_{u\ell}^j \, f_{u\ell} \ .$$

where we used $f_{u\ell}^j := f(K_\ell^j, K_u^{j-1})$. Let $\phi_{u\ell} := \mu_u^{j-1} f_{u\ell}^j$, then we can write the above equation in matrix notation conveniently as

$$\mu^{j-1\prime} \, \Pi^j \, f_{u\ell}^j = \Pi^{j\prime} \left(\mu^{j-1} \ f_{u\ell}^j\right) = \Pi^{j\prime} \phi_f \ .$$

Considering both $\Pi^j \equiv \kappa^j$ and $\phi_f \equiv \varphi_f$ as vectors, we see that the price of f under Π is given by

$$\pi^j(f) = \varphi_f' \kappa^j \ ,$$

which is once more just a linear equation in terms of κ^j. Hence, a set $\mathcal{F}^j$ of functions f for each maturity τ_j $(j > 1)$ yields, for each j, an equation of the type

$$V^j \kappa^j = \pi^j \ .$$

Now assume we have "weak information" in the form of some estimated market prices $\tilde{\pi}^j$. Then, we can formulate

Conclusion D.7 *The* weakly constrained *expensive martingale kernels* Π^j *are given as the solutions to the optimization problems*

$$\begin{cases} \textit{minimize} & ||V^j \kappa^j - \tilde{\pi}^j|| \\ \textit{such that} & M^j \kappa = z^j \\ & \kappa \geq 0 \end{cases} \tag{D.5}$$

for $M^j \in \mathbb{R}^{D^j \times N^j}$ *and* $V^j \in \mathbb{R}^{R^j \times N^j}$ *where* R^j *is the number of "weak information prices" at* τ_j*.*

Solutions to (D.5) can be found with straight-forward linear programming in case $||x|| := ||x||_\infty$ or $||x|| := ||x||_1$. In the more natural case $||x|| := ||x||_2$, we obtain a constrained linear least-squares programming problem, which can also be solved efficiently, see for example Fang et al. [FP93]. The choice of a norm (which we could also equip with some additional weighting) indicates how we see our "weak information".

The resulting kernels can be used to price exotic options. Some straight-forward details on this can be found in [B06a].

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