Quantitative Research GED

Valuing and Hedging Equity Derivatives.

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Valuing and Hedging Equity Derivatives

Overview

- Introduction
- Local Volatility Calibration via Forward PDEs
  - Standard case
  - Incorporating stochastic rates, default risk etc.
- Pricing and Hedging Option on realized Variance
  - Stochastic Volatility
  - Fitting the variance swap curve
  - Hedging
  - Term-structure models

Equity derivatives

Challenges

- A good fit to today's implied volatility surface.
- For Hybrid trades, we need to take into account additional risk factors:
  - Stochastic interest rates, default risk, dividends,…
- For "Volatility products", it is important to use models which generate reasonable future shapes of implied volatility
  - Options on Variance, Cliquets

Beyond Black-Scholes
Issues

- To address the deficiencies of the equity part in Black-Scholes' model, three basic components are at our disposal:
  - *Local Volatility* (ie, a spot-level dependent volatility function)
  - *Stochastic Volatility* (ie, volatility is a stochastic process on its own)
  - *Jump processes* (→ Rama Cont's talk this morning)
  - Mixtures of the above

Beyond Black-Scholes

Framework

- We denote by r the short rate and by d the dividend yield of the stock. Moreover, we assume the stock price defaults to zero at an inaccessible exponentially distributed default time τ with intensity λ.

  The stock price is then given as

  $$dS_t = S_t(r_t + \lambda_t - d_t)dt + S_t \frac{dX_t}{X_t} - S_t d1_{t \geq \tau}$$

  for some positive martingale X, which we will call the "pure" stock price.

- We will also use the symbols

$$DF_T = e^{-\int_0^T r_s ds}, \quad SV_T = e^{-\int_0^T \lambda_s ds}, \quad F_T = S_0 e^{\int_0^T (r_s + \lambda_s - d_s) ds}$$

Local Volatility.

Repricing Europeans

Local Volatility

Dupire's formula

- The implied volatility surface can be fitted "perfectly" using an *implied local volatility* approach:

$$\frac{dX_t}{X_t} = \sigma_t(X_t)dW_t$$

- If r, λ and d are deterministic, Dupire's formula (1994) allows us to extract the function σ from an option price surface,

$$\sigma_t(x)^2 = 2\frac{\partial_T \widetilde{C}(T,x)}{x^2\,\partial^2_{xx}\widetilde{C}(T,x)}$$

Market call price for strike $K=xF_T$

where we make use of the "pure" call prices on X,

$$\widetilde{C}(T,x) := \mathrm{E}[\,(X_T - x)^+\,] = \frac{1}{DF_T SV_T F_T}C(T,xF_T)$$

- Theoretically very nice, but numerically difficult to use.

Local Volatility
Forward PDEs

- It works better to calibrate the local volatility using forward-PDEs
  - The density p of the "pure" process X is the solution to the Fokker-Planck equation

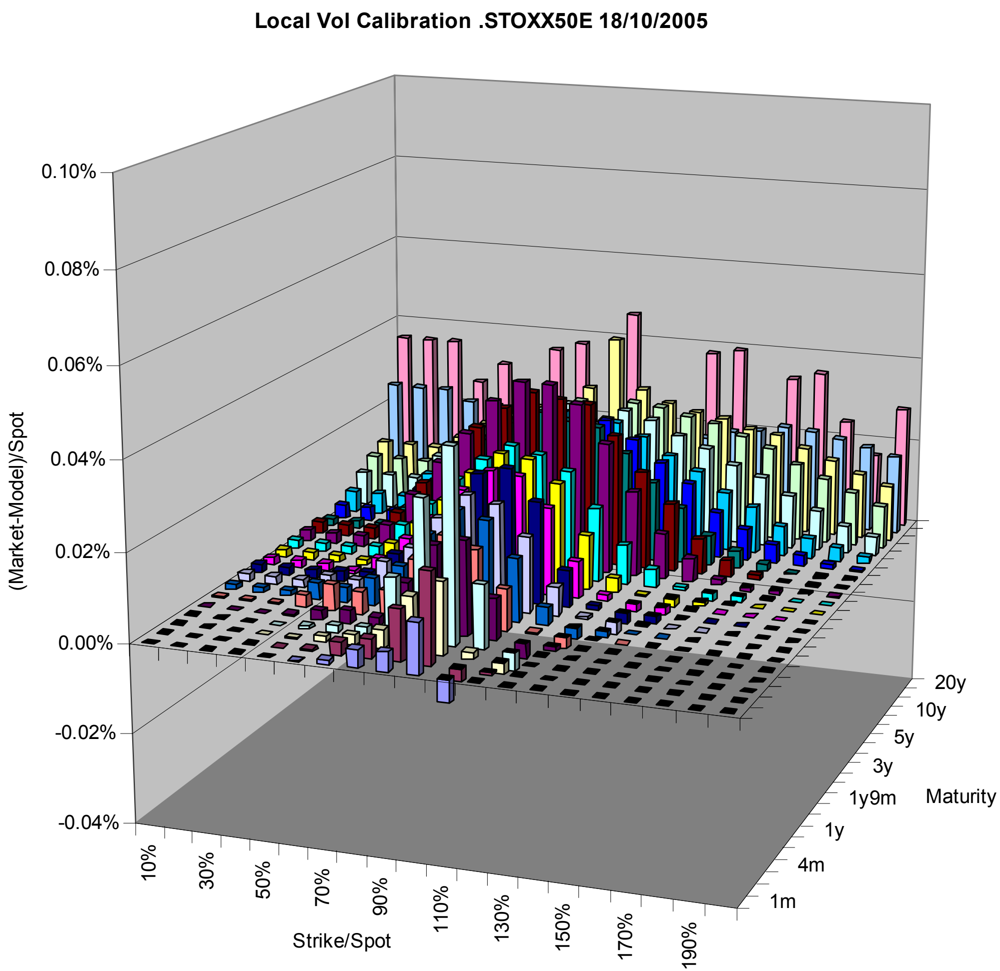
$$\partial_T p_T(x) = \tfrac{1}{2}\partial_{xx}^2\left\{\sigma_T(x)^2 x^2 p_T(x)\right\}$$

  - It can be written as the second derivative in strike of the "pure" call price, so we obtain

$$\partial_T \widetilde{C}(T,K) = \tfrac{1}{2}\partial_{KK}^2\left\{\sigma_T(K)^2 K^2 \widetilde{C}(T,K)\right\}$$

- This equation can easily be implemented on a finite-difference grid.
- Since it is a forward-equation, it allows the calibration of the local volatility function via bootstrapping.
  - Self-correcting scheme / linear growth in maturity / fast.
- For σ, we chose a bounded transformation of a cubic spline.
- Additional constraints on the second derivative of σ improve the stability of the algorithm.

Local Volatility
Examples



Local Volatility

Pricing Hybrids – stochastic interest rates

- To take into account stochastic interest rates, we can use a similar approach as before.
  - For example, assume we have calibrated a short-rate process

$$dr_t = a_t(r_t)dt + b_t(r_t)d\hat{W}_t \qquad \hat{W}_t = \rho W_t + \sqrt{1-\rho^2}W_t^{\perp}$$

  - We assume that the marginal density of r is continuous over $(-\infty,+\infty)$ (otherwise, boundary conditions need to be taken into account).
  - Denote by

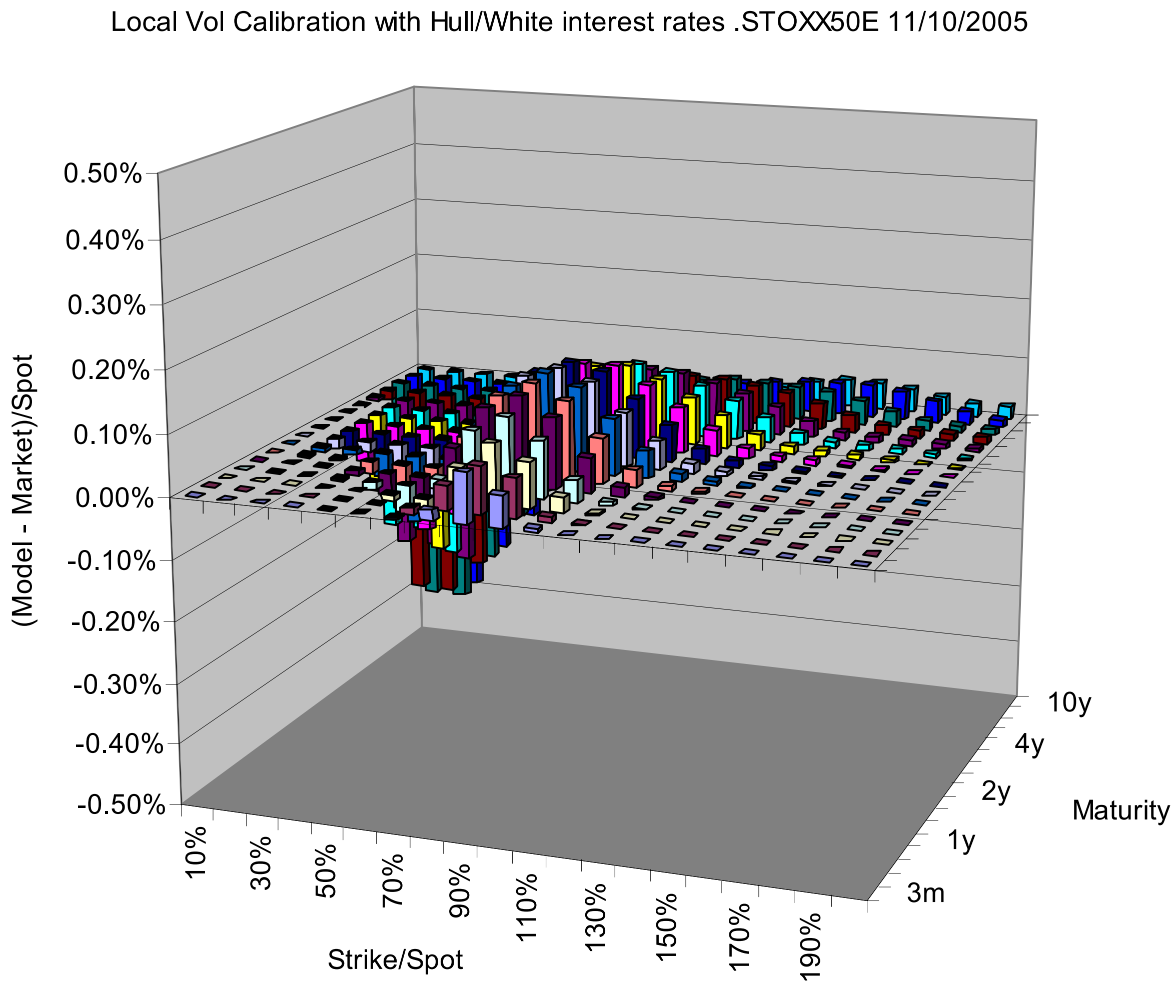
$$S_T^* = F_T X_T$$

  the stock process conditional on survival ($\tau > T$).

- The "non-default" density p of (S^*_T, r_T) under the T-forward measure is the solution to

$$\begin{aligned}\partial_T p_T(x,r) = &\\ &-rp_T(x,r) - \partial_x\left\{(r+\lambda_T - d_T)xp_T(x,r)\right\} - \partial_r\left\{a_T(r)p_T(x,r)\right\}\\ &+ \tfrac{1}{2}\partial^2_{xx}\left\{\sigma_T(x)^2x^2p_T(x,r)\right\} + \tfrac{1}{2}\partial^2_{rr}\left\{b_T(r)^2p_T(x,r)\right\}\\ &+\rho\,\partial^2_{xr}\left\{b_T(r)\sigma_T(x)xp_T(x,r)\right\}\end{aligned}$$

Beyond Black-Scholes
Local volatility



<u>FD grid size:</u>
Mesh 1500/500
Steps 100

Local Volatility
Pricing Hybrids – stochastic interest rates

- This forward equation can be solved on a two-factor grid
  - Robust scheme
  - The same procedure works (with a very similar PDE) for a stochastic intensity process or a stochastic dividend yield.
  - Good work-horse for one- or multi-factor hybrid products.

- For more details, see “Equity Hybrid Derivatives” (2006)

Local Volatility
Upside

- Excellent fit to European market prices.

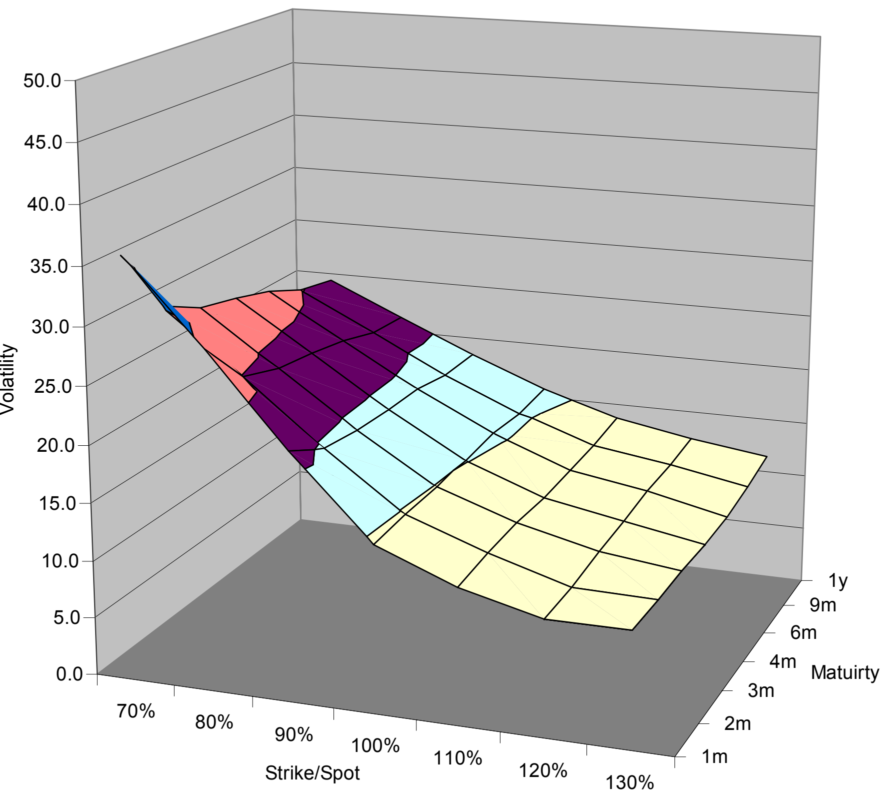
- Incorporation of “secondary” risk factors possible.

- Hedging
  - In theory, pure delta-hedging sufficient to hedge a product.
    The delta in local volatility takes into account the volatility skew.
  - In practice additional Vega-hedging with European options necessary.
    This is an “external” hedge whose costs are not covered by the initial price.
  - But performs well for trades which are not massively dependent on volatility.

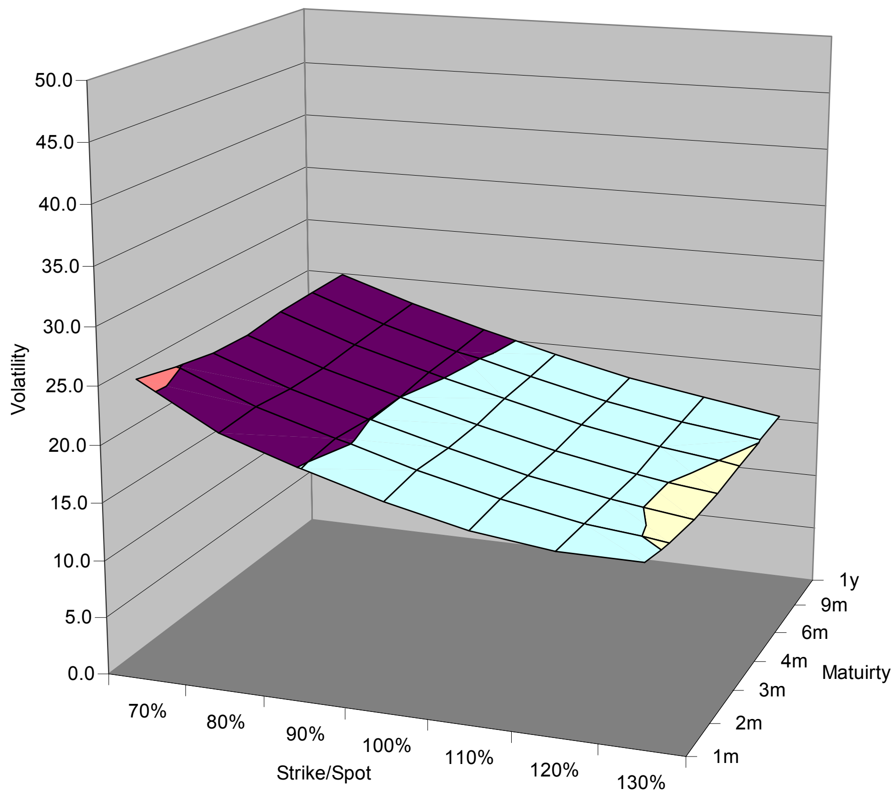
Local Volatility

Downside

Local Vol Calibration .STOXX50E 10/10/2005



Local Vol Calibration .STOXX50E 10/10/2005 shifted forward by 2 years



Local volatility shifted ahead for five years, same spot level.

Local Volatility

Downside

- Problems
  - Unrealistic future scenarios of implied volatility surface.
  - Very strong dependency on the stock level
- Not suitable for
  - Forward started options (which depend on future skew) and derivations thereof.
  - Options on realized variance, where a concept of “VolOfVol” is required.

- Need for models which preserve the general shape of implied volatility in a non-deterministic way.
  $\rightarrow$ Stochastic Volatility

Stochastic Volatility Models.

Stochastic Volatility

Concept

- Traditional stochastic volatility models use

$$\frac{dX_t}{X_t} = \sigma_t dW_t$$

with a *stochastic process* σ, usually driven by a one-dimensional SDE

$$\begin{aligned} \sigma_t &= f(v_t) \\ dv_t &= a_t(v_t)dt + b_t(v_t)d\hat{W}_t \qquad v_0 \in \mathbb{R} \\ \hat{W}_t &= \rho W_t + \sqrt{1-\rho^2}\, W_t^{\perp} \end{aligned}$$

VolOfVol function

Drift

Current state

Correlation

  – Given only the stock price, such a model is incomplete.
  – With one additional instrument, it becomes complete .
  – Correlation ρ governs the degree of dependency between variance and spot, but this dependency is weaker than in local volatility.

Stochastic Volatility
Heston

- Standard Example: Heston (1993)

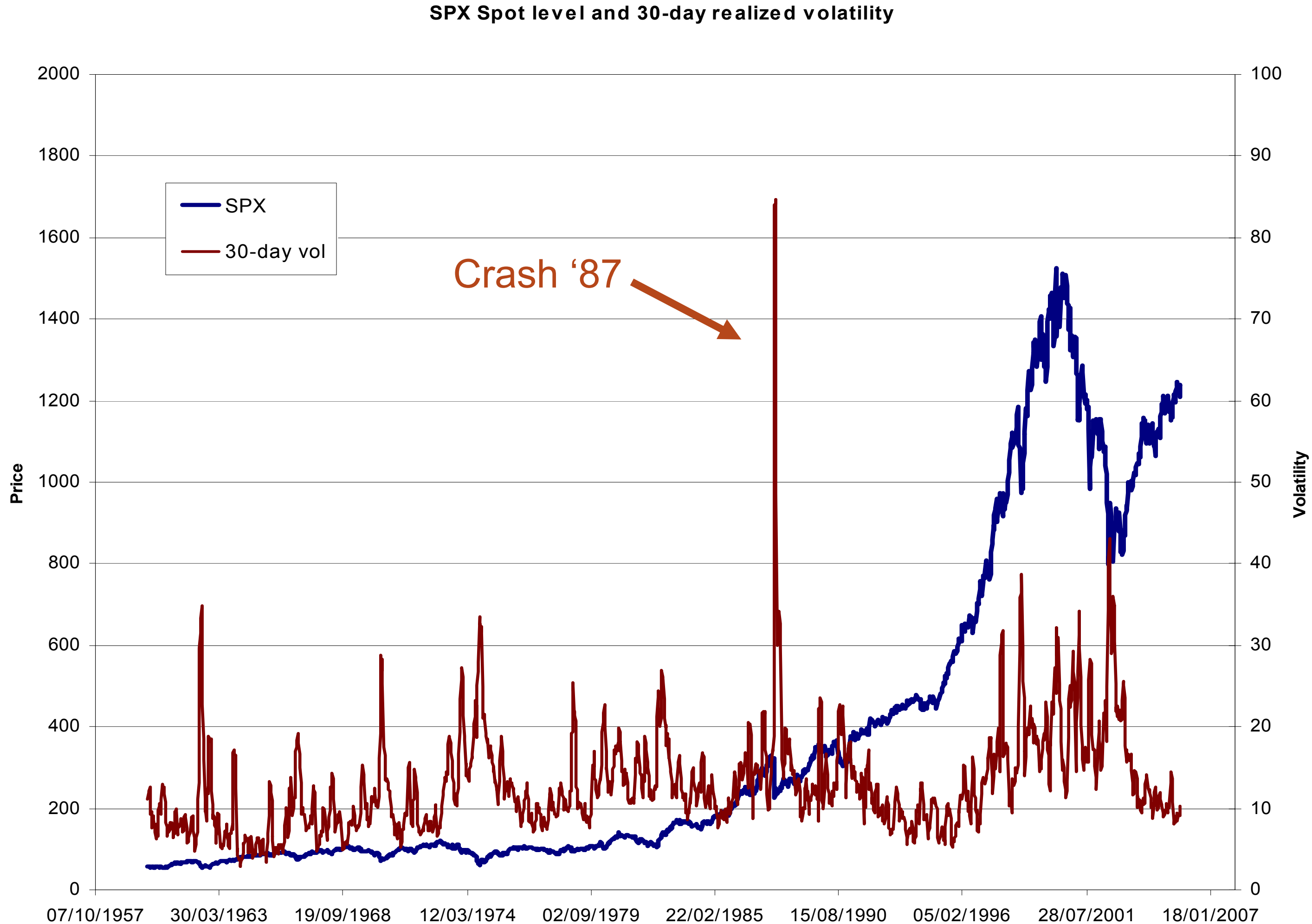
$$\sigma_t = \sqrt{v_t}$$
$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}\,d\hat{W}_t \qquad v_0 > 0,\ \rho < 0$$

Reversion Speed (κ), LongVar (θ), VolOfVol (ξ), ShortVar (v_0), Correlation (ρ)

- – Intuitive description of the short variance v as a mean-reverting process.
- – Square-root mainly a matter of convenience:
  - Characteristic function can be computed, so various numerical methods can be applied. In particular, we can compute European option prices using Carr/Madan (1999)
  - The process does not become negative. However, it can reach zero if $2\kappa\theta/\xi^2 \leq 1$ (which is often the case if the model is freely calibrated to market data).

Stochastic Volatility

Why mean-reversion?



Stochastic Volatility
Other examples

- More examples

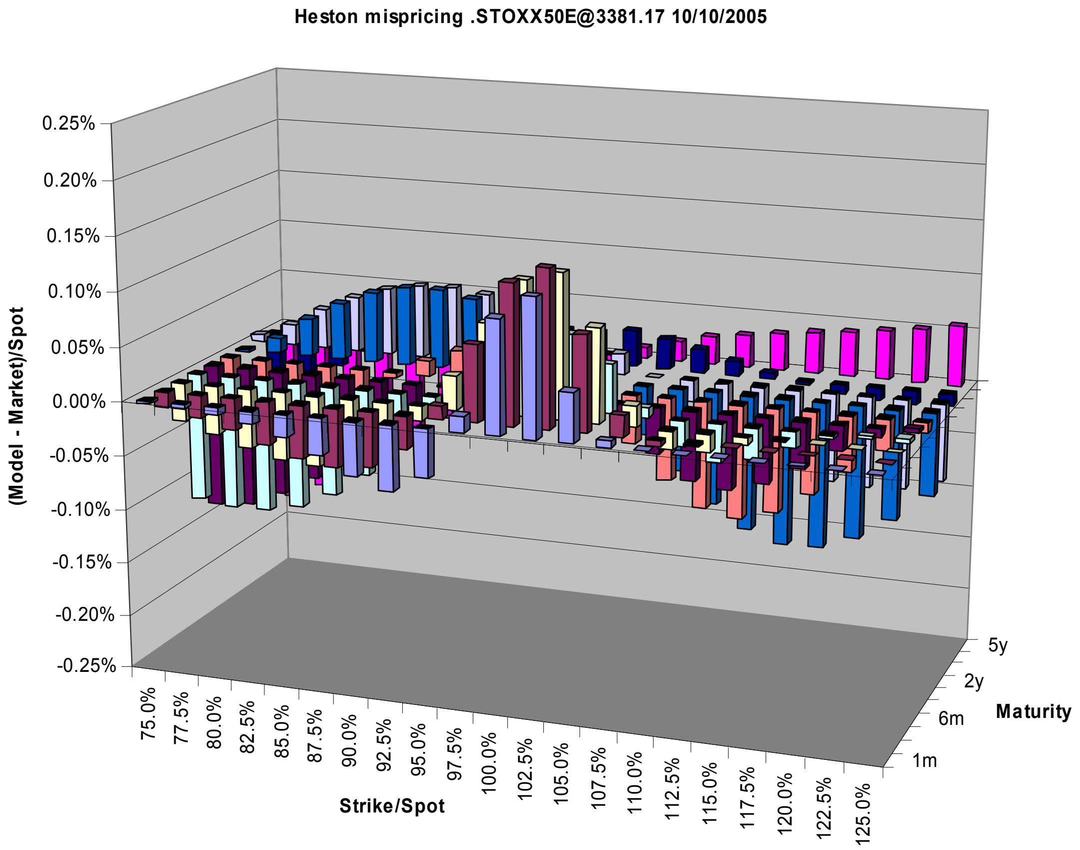
$$
\begin{aligned}
dv_t &= \kappa(\theta - v_t)dt + \xi v_t^{\alpha} d\hat{W} & \sigma_t &= \sqrt{v_t} & &\text{Generalized Heston} \\
dv_t &= \kappa(\theta - \sigma_t)dt + \xi d\hat{W} & \sigma_t &= v_t & &\text{Stein \& Stein 1991} \\
dv_t &= \eta v_t dt + \xi v_t d\hat{W} & \sigma_t &= \sqrt{v_t} & &\text{Hull \& White 1987} \\
dv_t &= \kappa(\theta - v_t)dt + \xi d\hat{W} & \sigma_t &= e^{v_t} & &\text{Scott 1987}
\end{aligned}
$$

  – Of the above only Stein&Stein allows the computation of European option pricing with Fourier-methods (Schoebel/Zhou 1999).

  – For all others, calibration via MC/FD/approximations is required.

Stochastic Volatility

Heston



Stochastic Volatility
Time-dependency

- In principle the coefficients a and b in the general form

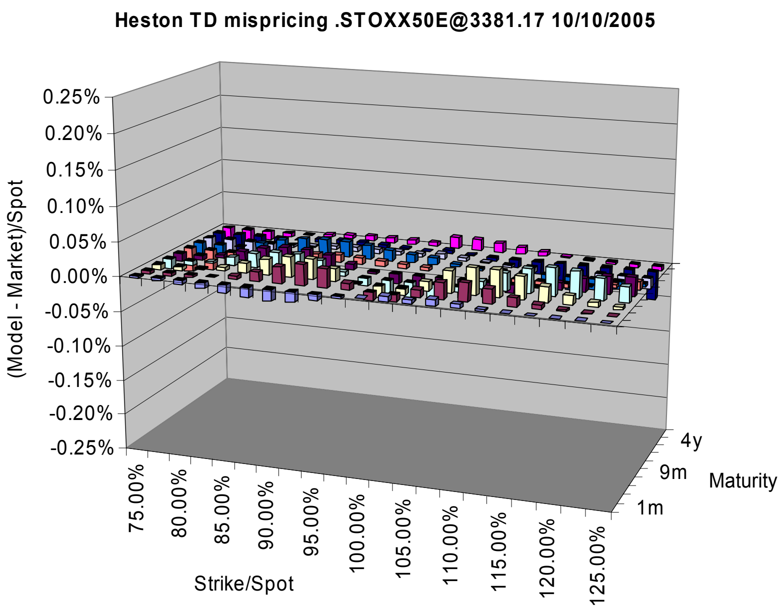
$$dv_t = a_t(v_t)dt + b_t(v_t)d\hat{W}_t$$

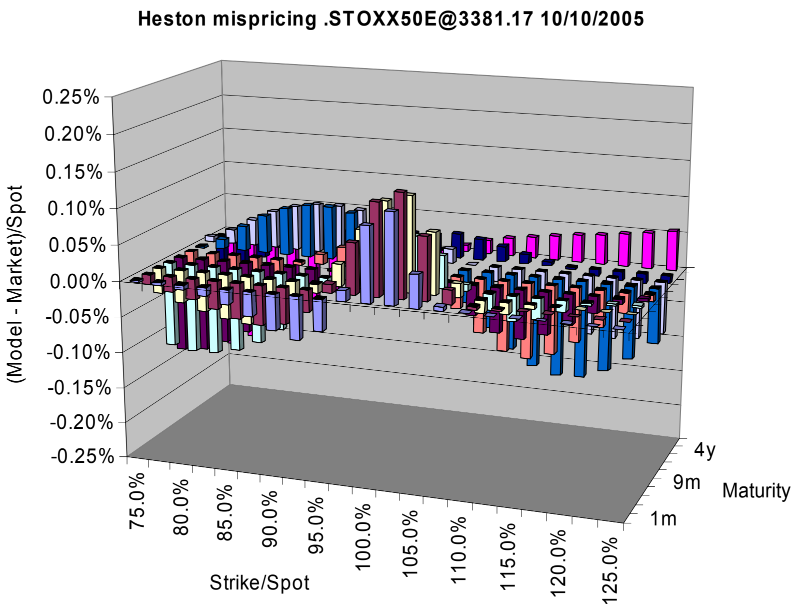
can also be time-dependent

– For Heston, piece-wise constant parameters are tractable and yield a very good fit to observed European market prices.

– For all others, bootstrapping via FD is feasible.

Stochastic Volatility
Heston: Time-dependent parameters

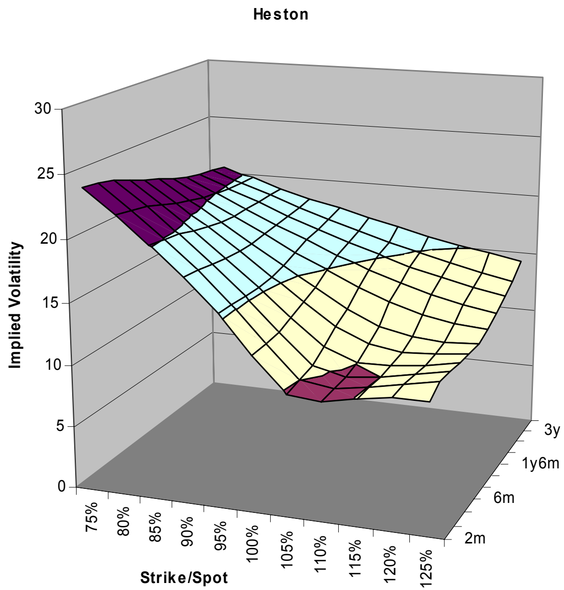




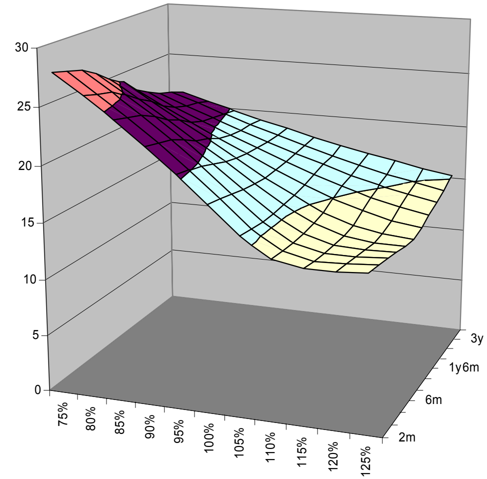
Left side:
piece-wise constant parameters for 3m, 1y, 3y and thereafter

Stochastic Volatility

Advantage of time-homogeneity

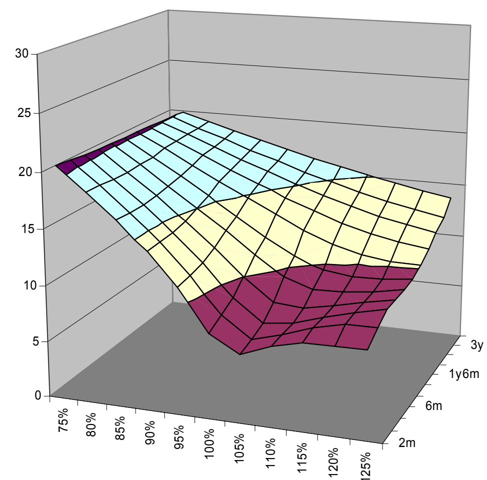


ShortVol +50%



The short end of the implied vol varies markedly, the long end is more stable.

ShortVol -50%



Unconstrained Calibration

ShortVol	12.4%
LongVol	21.2%
RevSpeed	1.17
Correlation	-0.74
VolOfVol	49.1%

Stochastic Volatility

Time-homogeneity vs. fitting

- Time-homogeneous models have “predictive power”: the possible shapes of future implied volatility surfaces in the model are given in terms of a known function of the state parameters.
  - If the parameters of a stochastic volatility model are constant, then the vector (S,v) is a time-homogeneous Markov-process.
  - This is a common property of stochastic volatility and jump models.
- Time-dependency destroys this property at least partly but improves today’s fit to the implied volatility surface.

Stochastic Volatility
Summary

- Pros
  - Future shapes of implied volatility are given as a known function of the random short variance.
  - Parsimonious structure.
  - Theoretically complete markets; a concept of vega-hedging is embedded in the model (see also next section)

- Cons
  - Relatively complicated numerics (apart from Heston's model).
  - Fit can be poor, but can be improved significantly by introducing modest time-dependency in the parameters
    - Mixing with local volatility is also possible, but we do not discuss this here.

Options on realized Variance.

Hedging with Variance Swaps

Options on realized Variance

Introduction

- Here, *realized variance* will be understood as

$$\langle \log S \rangle_T$$

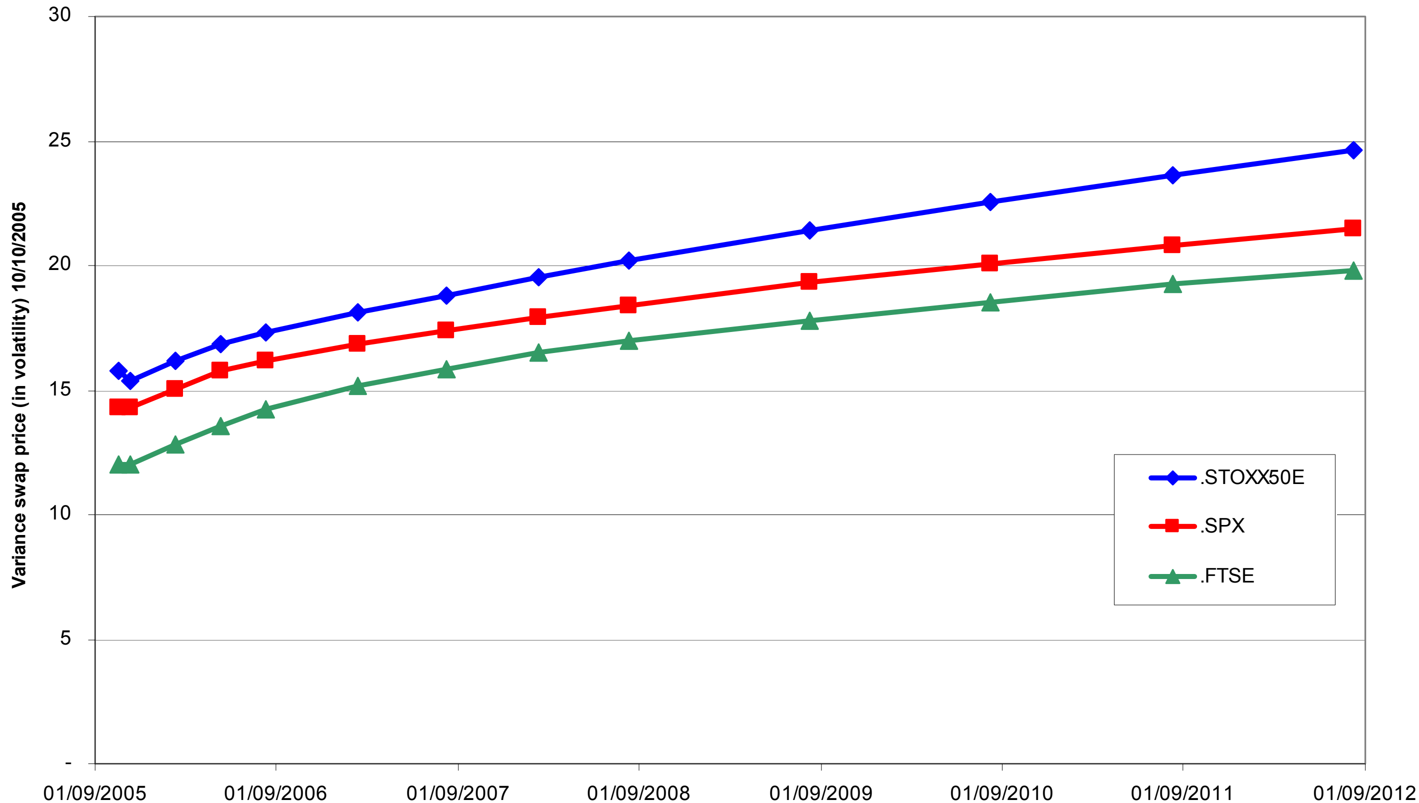
- A *variance swap* pays out the (annualized) realized variance in exchange for a previously agreed fixed amount K^2:

$$\frac{1}{T}\langle \log S \rangle_T - K^2$$

  - In the sequel, we will assume that interest rates and dividend yield are deterministic and that the stock cannot default.
    This means in effect that we can assume that $r=d=\lambda=0$.
  - If the stock price is continuous and if European options are traded for all strikes, then the price of the variance swap is given uniquely in terms of European option prices (see Demeterfi *et al* 1999).

Options on realized Variance

Variance Swap Markets



Options on realized Variance

Introduction

- Various vanilla options are now traded OTC
  - Calls and puts

$$\left(\tfrac{1}{T} \langle \log S \rangle_T - K^2 \right)^+ \qquad \left(K^2 - \tfrac{1}{T} \langle \log S \rangle_T \right)^+$$

  - Volatility Swaps

$$\sqrt{\tfrac{1}{T} \langle \log S \rangle_T} - K$$

- A variant of variance swaps are *weighted variance swaps* or *gamma swaps*

$$\tfrac{1}{T} \int_0^T \frac{S_t}{S_0} d\langle \log S \rangle_t - K^2$$

$\rightarrow$ options thereon.

Options on realized Variance

Using variance swaps as primary instruments

- Since variance swaps are now reasonably liquid, the idea is to use them as primary hedging instruments for volatility products.
  - All models considered here are stochastic-volatility type models (no jumps and no local volatility component).

- Two approaches
  - Fitting stochastic volatility models
  - Variance Curve term-structure models

Fitting the Variance Swap Curve.

Options on realized Variance

Fitting stochastic volatility

- A group of stochastic volatility models which match perfectly the initial variance swap curve $V_0(T),\ T\geq 0$ can be constructed as follows:
  - Assume we are given a stochastic volatility model

$$\frac{d\tilde{X}_t}{\tilde{X}_t} = \sqrt{\tilde{v}_t}\,dW_t \qquad d\tilde{v}_t = a(\tilde{v}_t)dt + b(\tilde{v}_t)d\hat{W}_t$$

for which we can compute the price of a variance swap,

$$\tilde{V}_0(T) = \mathrm{E}\left[\left\langle \log \tilde{X} \right\rangle_T\right] = \mathrm{E}\left[\int_0^T \tilde{v}_s ds\right] = \int_0^T \mathrm{E}[\tilde{v}_s]ds = \int_0^T \tilde{u}_0(s)ds$$

with initial *forward variance*

$$\tilde{u}_0(t) := \mathrm{E}\left[\tilde{v}_t\right].$$

Options on realized Variance

Fitting stochastic volatility

- Then let

$$\frac{dX_t}{X_t} = \sqrt{v_t}\,dW_t \qquad v_t := \eta_t \tilde{v}_t \qquad \eta_t := \frac{\partial_T V_0(t)}{\tilde{u}_0(t)}_t$$

- Examples:
  - The first model of this kind to our knowledge was proposed by Dupire (1992), which is based on an log-normal short variance process (cf. Hull-White model).

$$d\tilde{v}_t = \xi\,\tilde{v}_t\,d\hat{W}_t$$

  - Closely related is Scott's stochastic volatility model (exponential-OU).

$$d\log(\tilde{v}_t) = -\kappa\log(\tilde{v}_t)dt + \xi d\hat{W}_t$$

Options on realized Variance

Fitting stochastic volatility

- It also works for Heston :

$$d\tilde{v}_t = \kappa(\theta - \tilde{v}_t)dt + \xi\sqrt{\tilde{v}_t}\,d\hat{W}_t$$

- An alternative for Heston-type models is

$$dv_t = \kappa(\theta(t) - v_t)dt + \xi\sqrt{v_t}\,d\hat{W}$$

$$\theta(t) := \partial_T V_0(t) + \tfrac{1}{\kappa}\partial^2_{TT} V_0(t)$$

$$v_0 := \partial_T V_0(0)$$

provided V_0 is sufficiently smooth and that θ is continuous and non-negative.

Options on realized Variance

Fitting stochastic volatility

- The remaining parameters of these models need to be calibrated to European options.
  - A “generic” approach is to bootstrap time-dependent parameters with a two-factor finite-difference engine (just as for our local volatility calibration).
  - For the alternative Heston approach above, note that

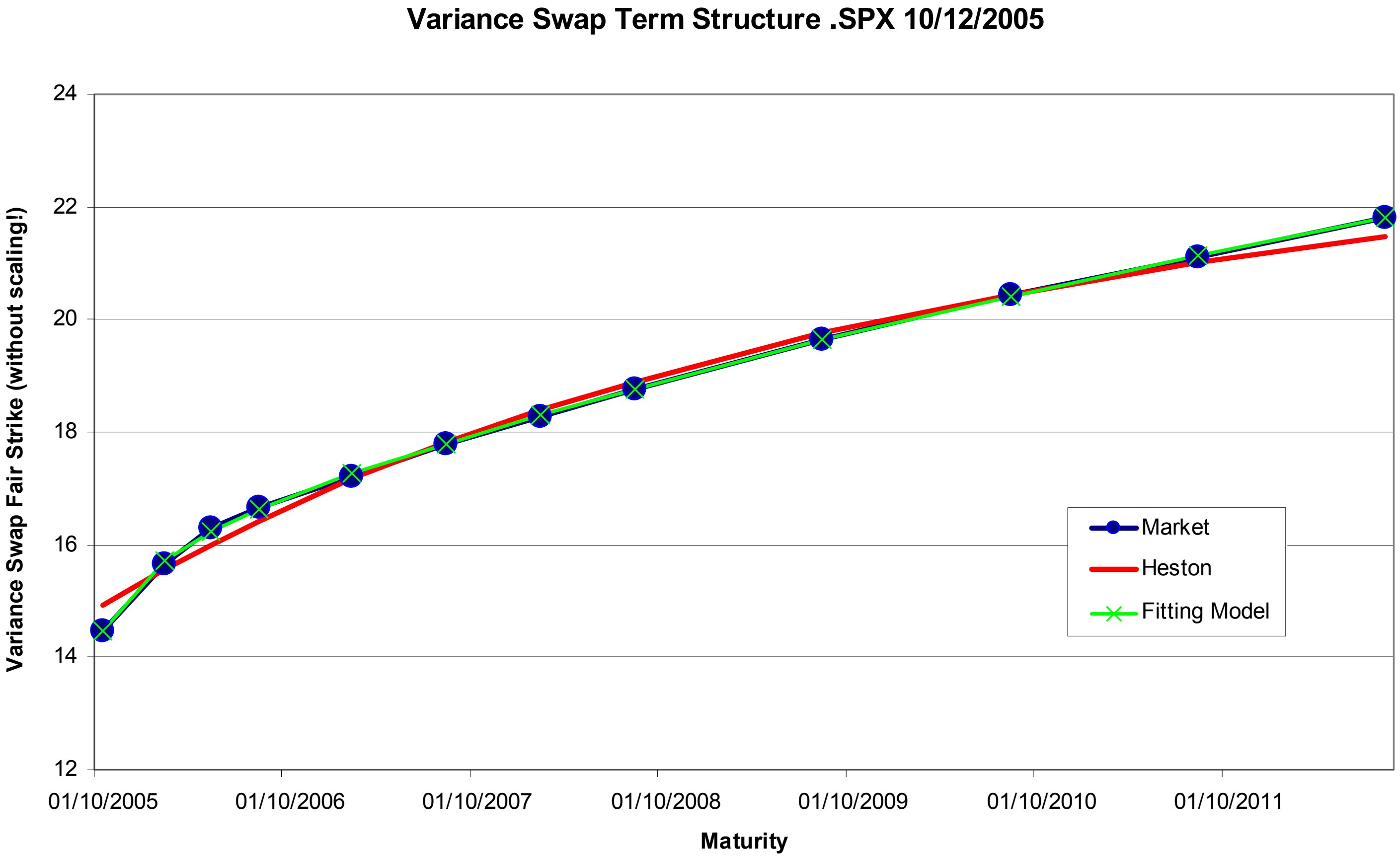
$$\mathrm{E}\left[\exp\left(iz\log X_t\right)\right]=\exp\left(-v_0\psi(z,t)-\int_0^t\kappa\theta(t-s)\psi(z,s)ds\right)$$

  where ψ is taken from the Heston-CF (cf. the case where θ constant). Hence, pricing and calibration via Fourier-inversion is feasible.

- The strategy is to fit a smooth function V_0 to the variance swap market data and then calibrate piecewise constant parameters (ρ,ξ) to European options.

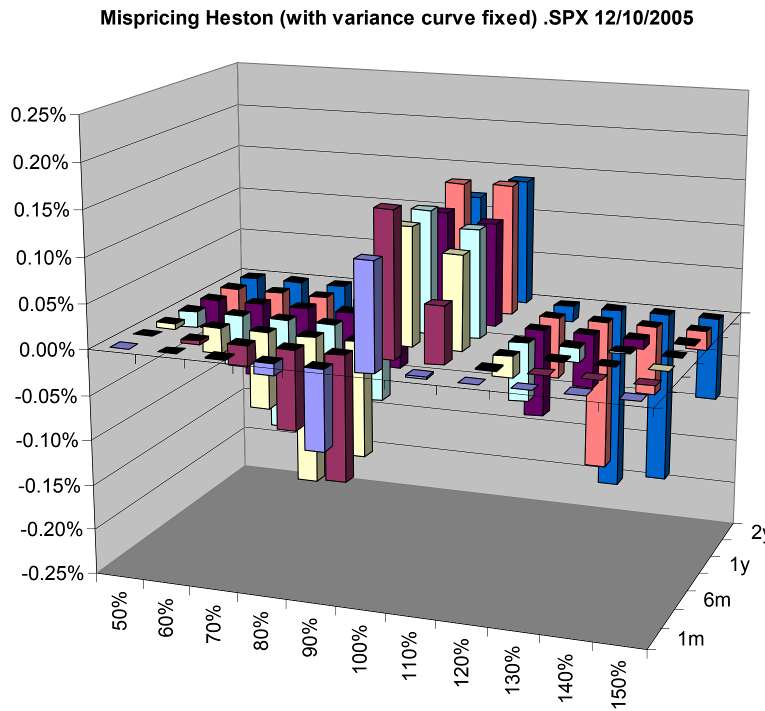
Options on realized Variance

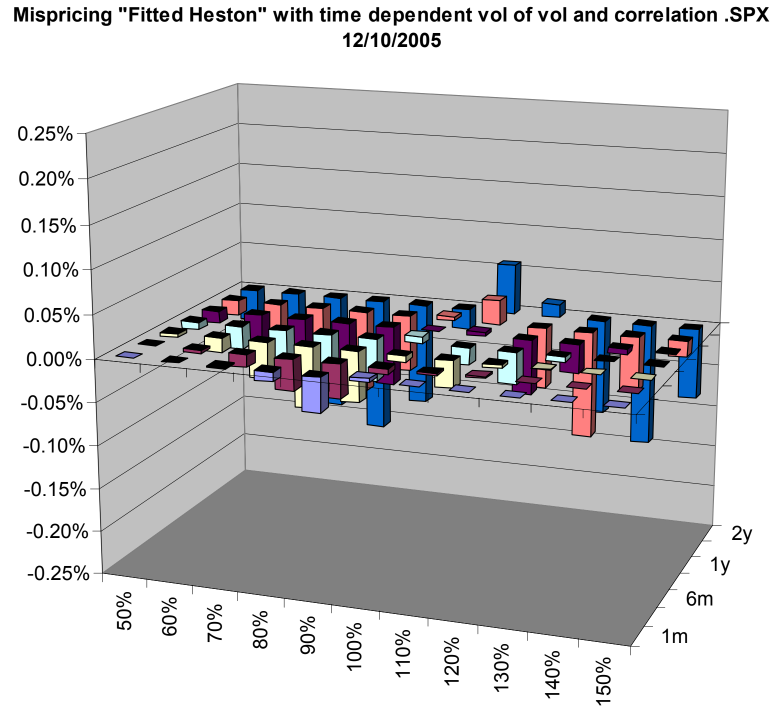
Fitting stochastic volatility



Options on realized Variance

Fitting stochastic volatility





Calibrate the Heston's variance curve first, then fit globla VolOfVol and Correlation.

Interpolate variance curve and then calibrate piece-wise flat VolOfVol and Correlation

Options on realized Variance

Hedging

- The price of a variance swap with maturity T at time t is given as

$$V_t(T) = \mathrm{E}\left[\int_0^T v_s ds \mid v_t\right] = V_t(t) + \Gamma_t(v_t; T-t)$$

Realized variance up to t — Remaining expected variance

where we define

$$\Gamma_t(v;x) := \mathrm{E}\left[\int_t^{t+x} v_s\, ds \mid v_t = v\right]$$

Assume that Γ is invertible and differentiable in v (weak assumptions on a and b are sufficient)

- Now we want to hedge an option with payoff

$$\mathrm{h}\left(S_T, \int_0^T v_s ds\right)$$

Options on realized Variance
Hedging

- By Markov-property of (S,v), the expectation of the payoff is a function

$$h_t(S_t, v_t, V_t(t)) := \mathrm{E}\left[\mathrm{h}\left(S_T, \int_0^T v_s ds \right) | \mathrm{F}_t \right]$$

hence, we can write it by inversion of Γ as

$$h_t(S_t, V_t(T), V_t(t)) \equiv h_t\left(S_t, \Gamma^{-1}(\cdot; T-t)(V_t(T) - V_t(t)); V_t(t)\right)$$

Variance swap with same maturity as the option.

- This yields the hedge

$$dh_t(\cdots) = \partial_s h_t(\cdots) dS_t + \partial_{V(T)} h_t(\cdots) dV_t(T)$$

"VarSwapDelta"

Options on realized Variance

Hedging: Example Call on Variance

- For Options on Variance such as calls

$$\mathrm{h}\left(S_T, \int_0^T v_s ds\right) := \left(\tfrac{1}{T} \int_0^T v_s ds - K^2 \right)^+$$

the option price depends only on the variance and not on S.
Hence, the option can in theory be replicated by hedging purely with the variance swap:

$$dh_t(\cdots) = \partial_{V(T)} h_t(\cdots) dV_t(T)$$

- Question:
  – What impact has the choice of a model on the “VarSwapDelta” ?

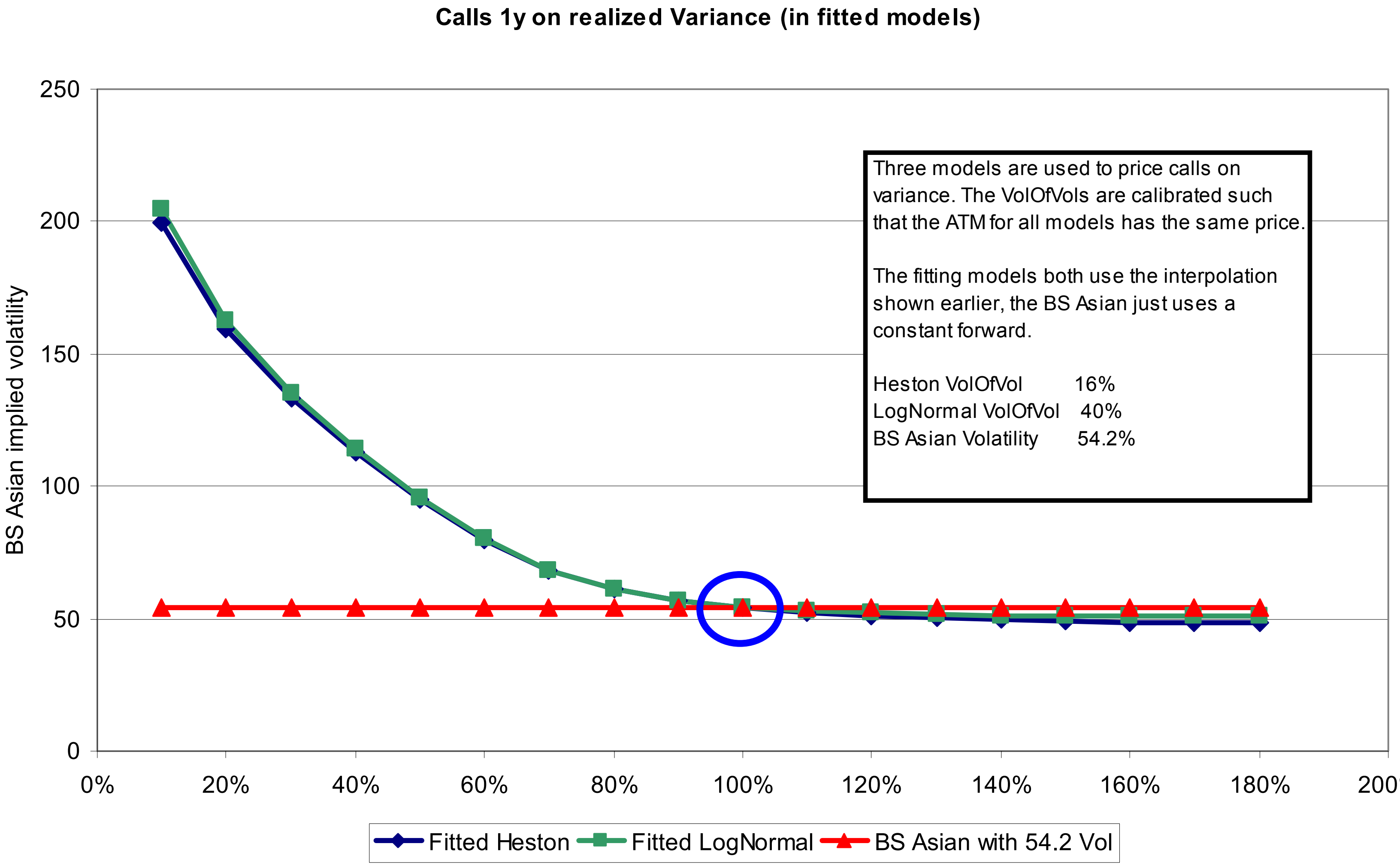
Options on realized Variance

Hedging: Example Call on Variance

- We compare a fitted Heston and a fitted Log-Normal variance model:
  - Calibrate them to the term structure of variance swaps.
  - Chose VolOfVol parameters such that the ATM 1y call on variance has the same price in both models.
  - Compare OTM option prices and VarSwapDeltas of the two models.
- As a reference, we quote prices in terms of “BS Asian” implied volatility (where we set the variance swap price as spot price of the “equity”).
  - The Vol in this model is also fitted to the ATM 1y.

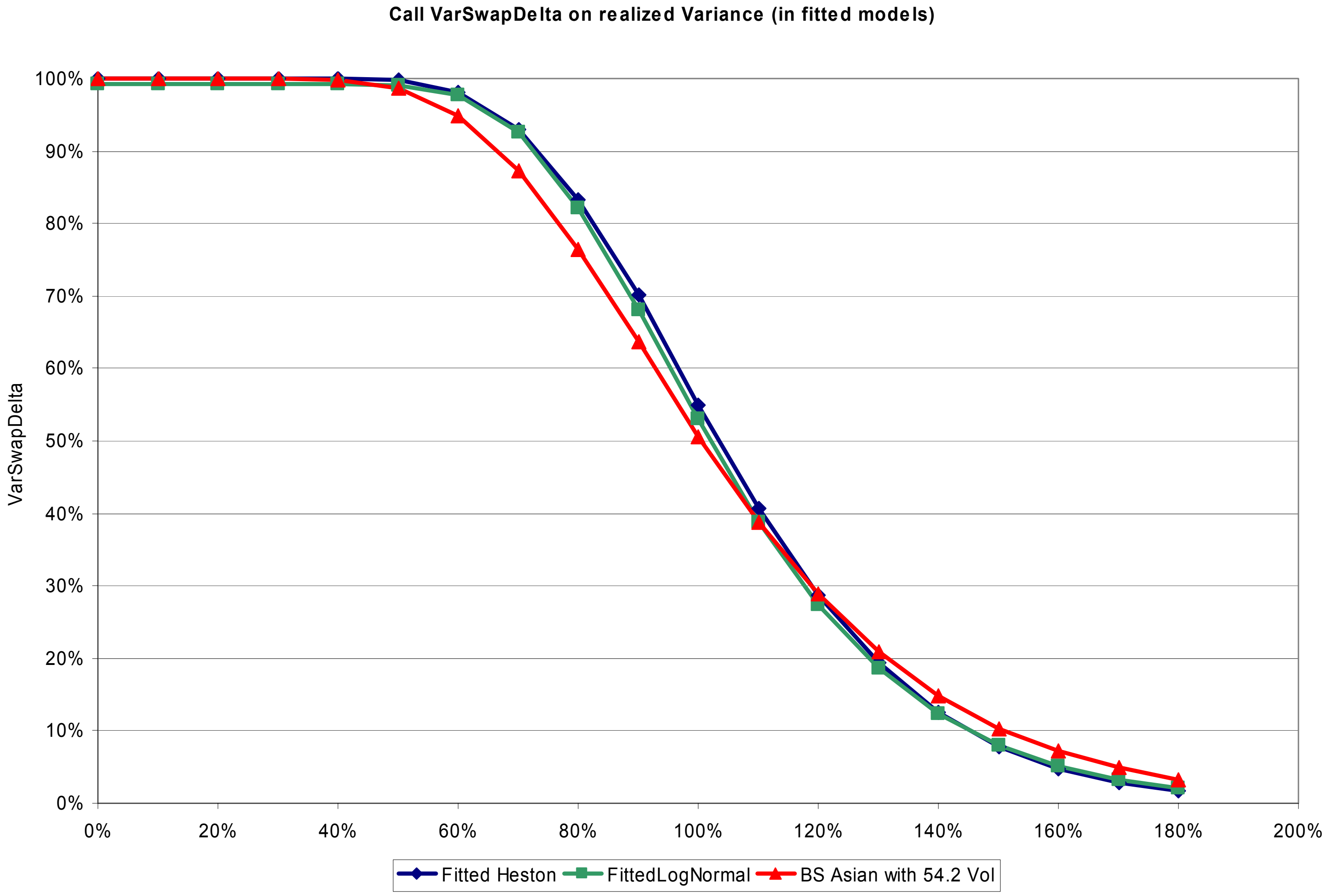
Options on realized Variance

Impact of the choice of a model: Price (quoted in BS Asian implied vol)



Options on realized Variance

Impact of the choice of a model: VarSwapDelta



Term-Structure approach.

Options on realized Variance

Term-Structure Models

- The previous approach described the evolution of the entire variance swap curve with one factor (the short variance).
- The next logical step is to model the entire forward variance curve

$$v_t(T) := (\partial_T V_t)(T)$$

as a *non-negative* C^2 functional G of an m-dimensional homogeneous diffusion Z ,

$$v_t(T) \equiv G(Z_t; T-t)$$

  – NB $G(z,x)$ is the *Musiela-parametrization* of v.
  – The diffusion Z is a unique strong solution to an SDE

$$dZ_t = \mu(Z_t)dt + \sum_{j=1}^{d} \sigma^j(Z_t)dW_t^j$$

where W is a d-dimensional Brownian motion.

Options on realized Variance

Term-Structure Models

- Using to the martingale-property of $v(T)=(v_t(T))_{0\leq t\leq T}$ we derive the consistency condition

$$\partial_x G(z;x) = \mu(z)\partial_z G(z;x) + \frac{1}{2}\sigma^2(z)\partial_{zz}^2 G(z;x)$$

  – We can use this equation to find coefficients (μ,σ) for a given function G.

- If G is a function of the form

$$G(z_1,\ldots,z_n,z_{n+1},\ldots,z_m;x) = \sum_{i=1}^n p_i(z;x)e^{-z_i x}$$

where p_i are polynomials, then $Z_1,\ldots Z_n$ must be constant for any consistent process.

Options on realized Variance

Term-Structure Models

- Once (G,Z) is determined, an associated stock price can be defined by

$$\frac{dX_t}{X_t} = \sqrt{G(Z_t;0)d}\sum_{j=1}^{d} \rho_j dW_t^j$$

  - The stock is a local martingale ($\rightarrow$ arbitrage-free market).
  - If ρ is a Lipschitz function of (S,Z), it also works. This allows to model local volatility and stochastic correlation in this framework.

- If G is "nicely" invertible, then the market of payoffs dependent on stock and variance swap prices (i.e., the economically relevant payoffs) is complete.
  - Each payoff can be hedged with stock and a finite number of variance swaps:

$$dh_t(\cdots) = \partial_s h_t(\cdots)dS_t + \sum_{i=1}^{M} \partial_{V(T_i)} h_t(\cdots)dV_t(T_i)$$

Options on realized Variance

Term-Structure Models

- A convenient example of the polynomial-exponential class is

$$G(z;x) = z_3 + (z_1 - z_2)e^{-\kappa x} + (z_2 - z_3)\frac{\kappa}{\kappa - c}\left(e^{-cx} - e^{-\kappa x}\right)$$

  - A consistent factor model for this G must have the form

$$\begin{aligned} dZ_t^1 &= \kappa(Z_t^2 - Z_t^1)dt + \sigma_1(Z_t)dW_t \\ dZ_t^2 &= c(Z_t^3 - Z_t^2)dt + \sigma_2(Z_t)dW_t \\ dZ_t^3 &= \sigma_3(Z_t)dW_t \end{aligned}$$

    which we call “double mean-reverting”.
  - Quite a good fit for most indices (at least during the course of the last year).
  - “Predictive” approach: Variance swap curves are always of a given shape.
  - Once the variance curve is fixed, obtain the remaining parameters via Monte-Carlo calibration.

Options on realized Variance
Summary

- Fitting models
  - Quick and intuitive solution.
  - Gives a hedging-strategy in terms of a single variance swap.
  - Heston-version can easily be calibrated to European options.

- Term-structure models
  - General “predictive” approach.
  - More complicated numerics.
  - Non-trivial interdependency between forward rates.
    This is important if we want to price options on variance swaps etc.

Thank you very much for your attention.

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