

Variance Swap Market Models

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Consistent Variance Curve Models

Outline

- Introduction
 - Options on realized variance
- Variance Curve Models
 - General theory
 - Finite-dimensionally parameterized curves
 - Variance Curves in a Hilbert space
- Examples
- Hedging



Realized Variance

Trading volatility



Consistent Variance Curve Models

Introduction

- We are given an equity index (S&P, EuroSTOXX,...).
- Listed options and complex OTC products are traded on S .
 - “Volatility” drives the price of such options.
 - Can we also *trade* “volatility” ?
- Well, we can trade *realized variance*.
- It is typically computed over the business days $0=t_0<\dots<t_N=T$ using the estimator

$$V^N(T) := \sum_{i=1}^N \left(\log S_{t_i} - \log S_{t_{i-1}} \right)^2$$

(up to scaling). But we will assume that it is actually given as

$$V^N(T) \approx \langle \log S \rangle_T$$



Consistent Variance Curve Models

Introduction

- We will assume that S is continuous, that it pays no dividends and that the interest rates are zero. Hence, we may write it as

$$\begin{aligned} S_t &= \exp\left(X_t - \frac{1}{2} \langle X \rangle_t\right) \\ dX_t &= \sqrt{\zeta_t} dB_t \end{aligned}$$

on a stochastic base $(\Omega, \mathcal{P}, \mathcal{F})$

- The one-dimensional Brownian motion B is adapted to the filtration \mathcal{F} .
 - The *short variance* process ζ is a predictable, integrable and non-negative.
- Realized variance is then the non-negative quantity

$$\langle \log S \rangle_T = \int_0^T \zeta_s ds$$



Realized Variance

Variance Swaps

- The simplest product on realized variance is a *variance swap*.
- A variance swap is just a forward on realized variance:
 - At maturity T it pays the realized variance occurred during the life of the contract (usually in exchange for a previously agreed fixed strike K).
 - Such contracts are today liquidly traded on most major indices. In particular, their price processes are martingales under each equivalent martingale measure on $(\Omega, \mathcal{P}, \mathbb{F})$.
 - Let us assume that P is a martingale measure.
 - Then, the price $V_t(T)$ of a variance swap is just the expectation of the realized variance,

$$V_t(T) = \mathbb{E} \left[\int_0^T \zeta_s ds \mid \mathbb{F}_t \right]$$



Realized Variance

Variance Swaps

- If European options are traded for all strikes, the price of a variance swap can in theory be computed in terms of European options using Neuberger's (1990) formula,

$$\begin{aligned} V_0(T) &= 2 \mathbb{E} \left[- \int_0^T \sqrt{\zeta_s} dB_s + \frac{1}{2} \int_0^T \zeta_s ds \right] \\ &= 2 \mathbb{E} \left[S_T - 1 - \log S_T \right] \\ &= 2 \left\{ \int_0^1 \frac{1}{K^2} \text{Put}(T, K) dK + \int_1^\infty \frac{1}{K^2} \text{Call}(T, K) dK \right\} \end{aligned}$$

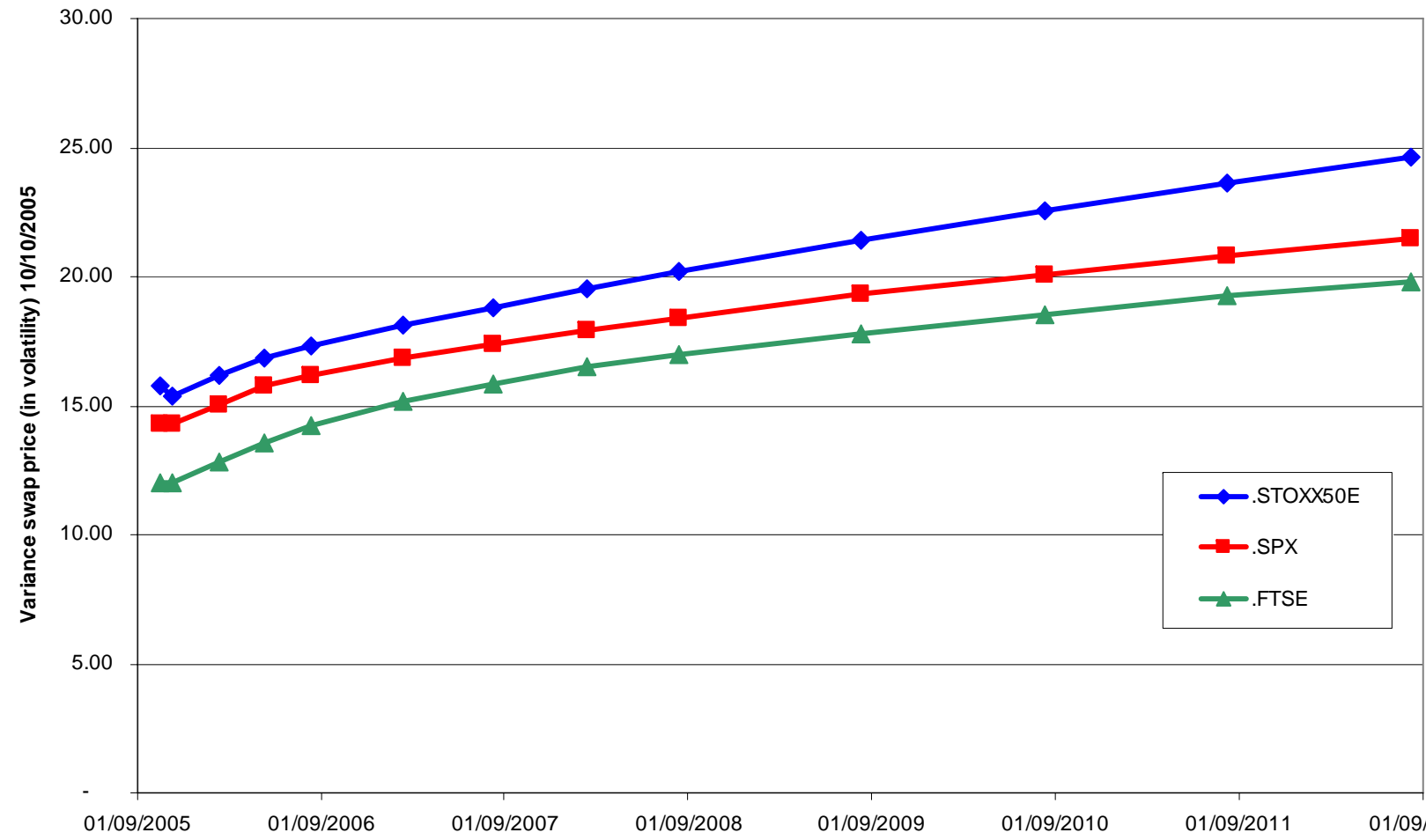
- This works only if option prices are available for all T .
- The formula probably contributes to the fact that variance swaps are now liquidly traded.
 - An excellent reference is Demeterfi et al (1999).



Realized Variance

Variance Swaps

Prices are quoted in "volatility" $\sqrt{\frac{1}{T} \int_0^T \zeta_s ds}$





Realized Variance

Beyond Variance Swaps

- Since variance swaps are liquidly traded, there is no need to price them.
- But what about more complex products:
 - *Calls* on realized variance

$$\left(\int_0^T \zeta_s ds - K^2 \right)^+$$

- *Volatility swaps*

$$\sqrt{\int_0^T \zeta_s ds} - K$$

- But also *forward started options*

$$\left(\frac{S_{T_2}}{S_{T_1}} - k \right)^+ = \left(\exp \left(\int_{T_1}^{T_2} \sqrt{\zeta_s} dB_s - \frac{1}{2} \int_{T_1}^{T_2} \zeta_s ds \right) - k \right)^+$$



Realized Variance

Beyond Variance Swaps

- The idea is to use the variance swaps themselves as underlying reference instruments.
 - A variance swap is a natural hedging instrument for such payoffs.
 - We can then attempt to hedge options on realized variance by delta-hedging with variance swaps.

- Mathematically, the term-structure of variance swaps reminds on the term-structure of discount bounds in interest rate models



Realized Variance

Forward Variance

- Variance swap prices are increasing with maturity T .
 - Their price at a later time t also depends on the past realized variance.
- To alleviate these unpleasant properties, note that

$$V_t(T) = \mathbb{E} \left[\int_0^T \zeta_s ds \mid \mathbb{F}_t \right] = \int_0^T \mathbb{E}[\zeta_s \mid \mathbb{F}_t] ds$$

can be differentiated in T to define the *forward variance*

$$v_t(T) := \partial_T V_t(T) = \mathbb{E}[\zeta_T \mid \mathbb{F}_t]$$

Observation time **Maturity**

- Note the similarity to the *forward rate* in interest rate theory.



Realized Variance

Modeling forward Variance

■ Idea

- Instead of starting with S , let us first specify the non-negative curve v .
- Then, construct a (local) martingale S which has the correct quadratic variation.
- The model shall also yield hedging ratios in terms of variance swaps.
- The main difference is that forward variance can be zero for good economic reasons (holidays, suspended trading).
 - Short variance also becomes zero for standard stochastic volatility models such as Heston's.



Variance Curve Models

A structural approach



Variance Curve Models

Classic approach

- First, we focus on the classical setup.
Assume we have a driving d -dimensional extremal Brownian motion W on the space $(\Omega, \mathcal{P}, \mathcal{F})$.

- Definition

A family $v = (v(T))_{T \geq 0}$ is called a *Variance Curve Model* if

1. For each $T > 0$, the process $v(T) = (v_t(T))_{t \in [0, T]}$ is a non-negative martingale:

$$dv_t(T) = \sum_{j=1, \dots, d} \beta_t^j(T) dW_t^j \quad \beta^j(T) \in L^{\text{loc}}$$

2. For each $T > 0$, the initial variance swap prices are finite, i.e.

$$V_0(T) = \int_0^T v_0(s) ds < \infty$$

3. The curve $v_t(t)$ is left-continuous.



Variance Curve Models

Classic approach

■ Properties

- The price processes of variance swaps,

$$V_t(T) := \int_0^T v_t(s) ds$$

are martingales.

- The *short variance process*

$$\zeta_t := v_t(t)$$

is well defined, integrable and non-negative.



Variance Curve Models

Classic approach

■ Properties

Given any standard Brownian motion B on $(\Omega, \mathcal{P}, \mathcal{F})$, the process

$$dX_t = \sqrt{\zeta_t} dB_t$$

is a square-integrable martingale, so the via B *associated stock price*

$$S_t := \exp\left(X_t - \frac{1}{2} \langle X \rangle_t\right)$$

is a local martingale.

- B represents the *correlation structure* of S with v .

■ Theorem

For each variance curve model v and each Brownian motion B , the market

$$\left(S; (V(T))_{T \geq 0}\right)$$

is free of arbitrage.



Variance Curve Models

Classic approach – Musiela-Parametrization

- As in interest rates, it is more convenient to work with fixed time-to-maturities $x := T - t$. Hence we define the *Musiela parameterization*

$$\hat{v}_t(x) := v_t(t + x)$$

- Starting in Musiela-parametrization

- Assume that $\sum_{j=1, \dots, d} \int_0^\infty \int_t^\infty \partial_T \beta_t(T)^2 dT dt < \infty$
Then,

$$d\hat{v}_t(x) := \partial_x \hat{v}_t(x) dt + \sum_{j=1, \dots, d} \hat{\beta}_t^j(x) dW_t^j$$

defines a variance curve model in Musiela-parametrization.



Variance Curve Models

Classic approach – Fitting the market

- If ν is represented as an exponential,

$$\hat{\nu}_t(x) := \exp(\hat{w}_t(x))$$

it allows us to fit the model easily to an observed market forward variance curve u_0 by setting $w_0 := \log u_0$.

- This construction does not allow ν to become zero.
- A more convenient approach is to use an existing curve ν^{base} and set

$$\hat{\nu}_t(x) := \frac{u_0(t+x)}{\hat{\nu}_0^{\text{base}}(t+x)} \hat{\nu}_t^{\text{base}}(x)$$



Variance Curve Models

Variance Curve Functionals

- Problems with a specification with general integrands $\beta(T)$:
 - It is complicated to check whether ν remains non-negative.
 - In practice, it is not clear how to handle such integrands computationally.
- Hence, we want to write

$$\hat{\nu}_t(x) := G(Z_t; x)$$

for some suitable non-negative function G and an m -dimensional Markov-process Z .



Variance Curve Models

Variance Curve Functionals

■ Definition

1. A non-negative $C^{2,2}$ -function $G: D \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is called a *Variance Curve Functional* if

$$\int_0^T G(z; x) dx < \infty$$

for all T and $z \in D$ where D is an open set in $\mathbb{R}_{\geq 0}^m$.

2. We denote by Ξ the set of all $C=(\mu, \sigma)$ for which the SDE

$$dZ_t = \mu(Z_t)dt + \sum_{j=1, \dots, d} \sigma^j(Z_t) dW_t^j$$

starting at any point $Z_0 \in D$ has a unique solution Z which stays in D .



Variance Curve Models

Variance Curve Functionals

- Definition

We call $C=(\mu, \sigma) \in \Xi$ a *consistent factor model* for G if for any $Z_0 \in D$,

$$\hat{v}_t(x) := G(Z_t; x)$$

defines a variance curve model.

- Theorem

This is the case if and only if Z stays in D and if

$$\partial_x G(z; x) = \mu(z) \partial_z G(z; x) + \frac{1}{2} \sigma^2(z) \partial_{zz}^2 G(z; x)$$

holds such that $G(Z_t; T-t)$ is a true martingale.



Variance Curve Models

Variance Curve Functionals

■ Remarks

- For each consistent pair (G, C) , we obviously have

$$G(z; x) \equiv \mathbb{E}[G(Z_x; 0) \mid Z_0 = z]$$

- But our interest was asked the reverse question:
given G , find a consistent $C = (\mu, \sigma) \in \Xi$.

- The next logical step is to model the curve v as a process with values in a Hilbert space H .
 - We follow the path laid by Bjoerk/Christensen (1999), Filipovic (2000), Filipovic/Teichmann (2004) and Teichmann (2005).



Variance Curve Models

Term-structure approach

- The main difference between variance curve and forward curves is that the curves v must remain non-negative (but *can* become zero).
 - The problem is that the “non-negative cone” is a very small set. Indeed, it has no interior points.
 - However, if $G(D)$ is a sub-manifold with boundary of H , then it is sufficient to check whether v stays in $G(D)$. In this case we say $G(D)$ is *locally invariant* for v .
 - If G is moreover invertible, we can also directly construct a (locally) consistent factor model $C=(\mu,\sigma)$ for G .



Variance Curve Models

Term-structure approach

- Assume that the variance curve v is given as a solution in H to

$$d\hat{v}_t = \partial_x \hat{v}_t dt + \sum_{j=1, \dots, d} \hat{\beta}^j(\hat{v}_t) dW_t^j$$

where the coefficients β are locally Lipschitz vector fields.

- The Stratonovic-drift for v is as usual

$$\beta^0(\hat{v}) := \partial_x \hat{v} - \sum_{j=1, \dots, d} D\beta^j(\hat{v}) \cdot \beta^j(\hat{v})$$

Frechet-Derivative

such that

$$d\hat{v}_t = \hat{\beta}^0(\hat{v}_t) dt + \sum_{j=1, \dots, d} \hat{\beta}^j(\hat{v}_t) \circ dW_t^j$$



Variance Curve Models

Term-structure approach

■ Theorem (Filipovic/Teichmann 2004)

The sub-manifold $G(D)$ is locally invariant for ν iff

1. We have $G(D) \subset \text{dom}(\partial x)$,
2. In the interior of $G(D)$, we have

$$\hat{\beta}^j(\hat{\nu}) \in T_{\hat{\nu}}G(D) \quad j = 0, \dots, d$$

3. On the boundary $\partial G(D)$,

$$\begin{aligned} \hat{\beta}^0(\hat{\nu}) &\in (T_{\hat{\nu}}G(D))_{\geq 0} \\ \hat{\beta}^j(\hat{\nu}) &\in T_{\hat{\nu}}\partial G(D) \quad j = 1, \dots, d \end{aligned}$$

holds.



Variance Curve Models

Term-structure approach

- If we can invert G , then $C=(\mu,\sigma)$ with

$$\sigma^j(z) := \partial_z G^{-1}(\hat{\beta}^j(G(z)))$$

$$\mu(z) := \partial_z G^{-1}(\hat{\beta}^0(G(z))) + \sum_{j=1,\dots,d} (\partial_z \sigma^j)(z) \sigma^j(z)$$

is (locally) a consistent factor model for G (the functions μ and σ are locally Lipschitz \rightarrow local existence).



Back to reality: A Simple Example

Linear mean-reversion



Variance Curve Models

Variance Curve Functionals – Linear mean-reversion

■ Example

A very basic example is the “linearly mean-reverting” functional:

$$G(z; x) = z_2 + (z_1 - z_2)e^{-z_3x}$$

„Long variance“ „Short variance“ „Speed of mean-reversion“

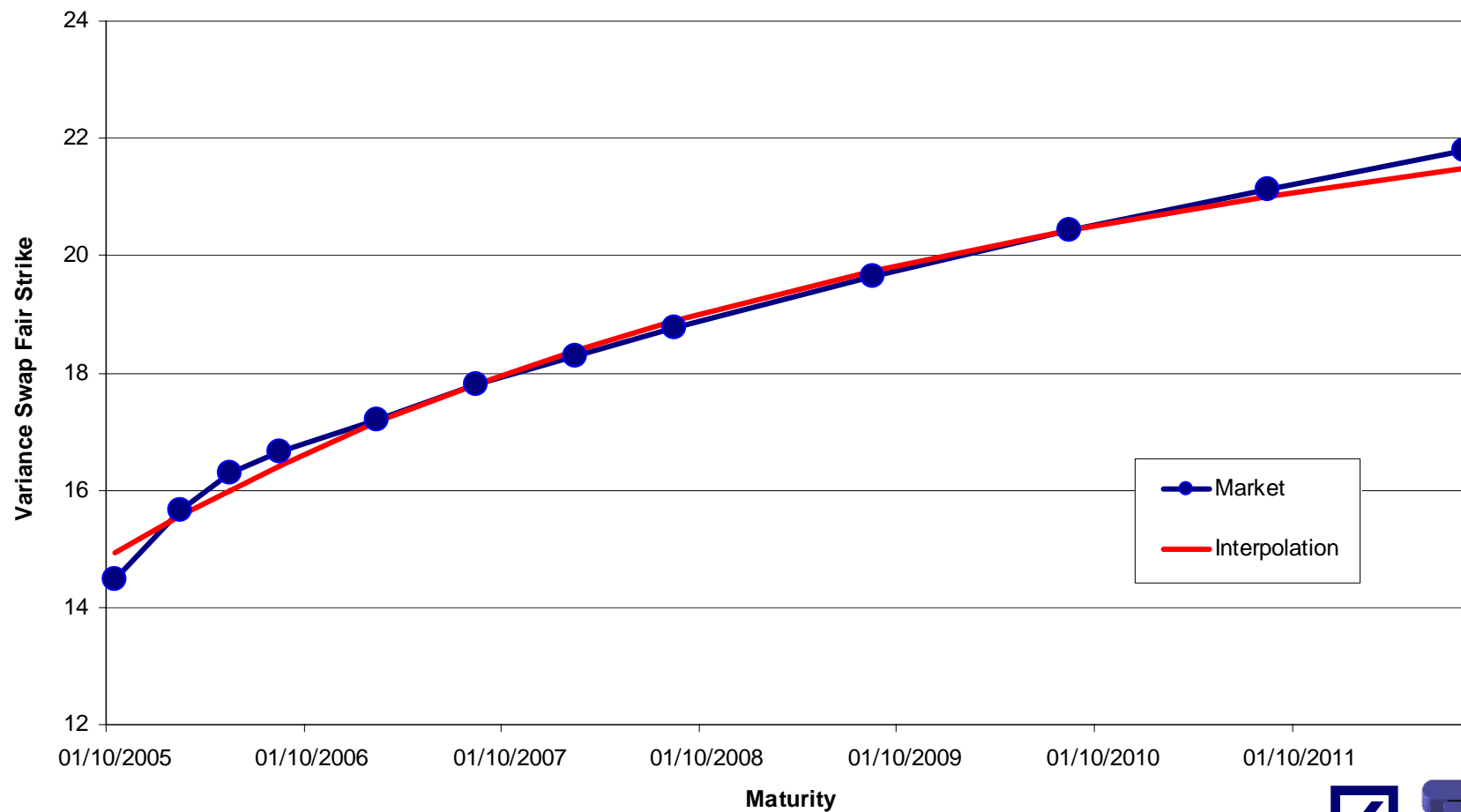
for $z_1 \geq 0$ and $z_2, z_3 > 0$.



Variance Curve Models

Variance Curve Functionals – Linear mean-reversion

Variance Swap Term Structure .SPX 10/12/2005





Variance Curve Models

Variance Curve Functionals – Linear mean-reversion

- Question: What dynamics can a consistent process $Z=(Z_1, Z_2, Z_3)$ have?
- The coefficients μ and σ have to satisfy

$$\partial_x G(z; x) = \mu(z) \partial_z G(z; x) + \frac{1}{2} \sigma^2(z) \partial_{zz}^2 G(z; x)$$

1. First, we see that

$$\partial_{z_3 z_3}^2 G(z; x) = (z_1 - z_2) x^2 e^{-z_3 x}$$

Since no term $x^2 e^x$ appears on the left hand side, we must have $\sigma_3=0$.

2. The same line of thought applied to

$$\partial_{z_3} G(z, x) = -(z_1 - z_2) x e^{-z_3 x}$$

shows that we also have $\mu_3=0$.

Hence, the speed of mean-reversion cannot be stochastic.



Variance Curve Models

Variance Curve Functionals – Linear mean-reversion

- For the other two parameters, we find that while σ is unconstrained,

$$\mu_2(z) = 0$$

$$\mu_1(z) = z_3(z_2 - z_1)$$

In other words: The only consistent processes for this choice of G are of Heston-type

$$d\zeta_t = \kappa(\theta_t - \zeta_t)dt + \sigma_1(\zeta_t, \theta_t)dW_t$$

$$d\theta_t = \sigma_2(\zeta_t, \theta_t)dW_t$$

Linear mean-reversion drift

VolOfVol can freely be chosen as long as ζ remains non-negative.

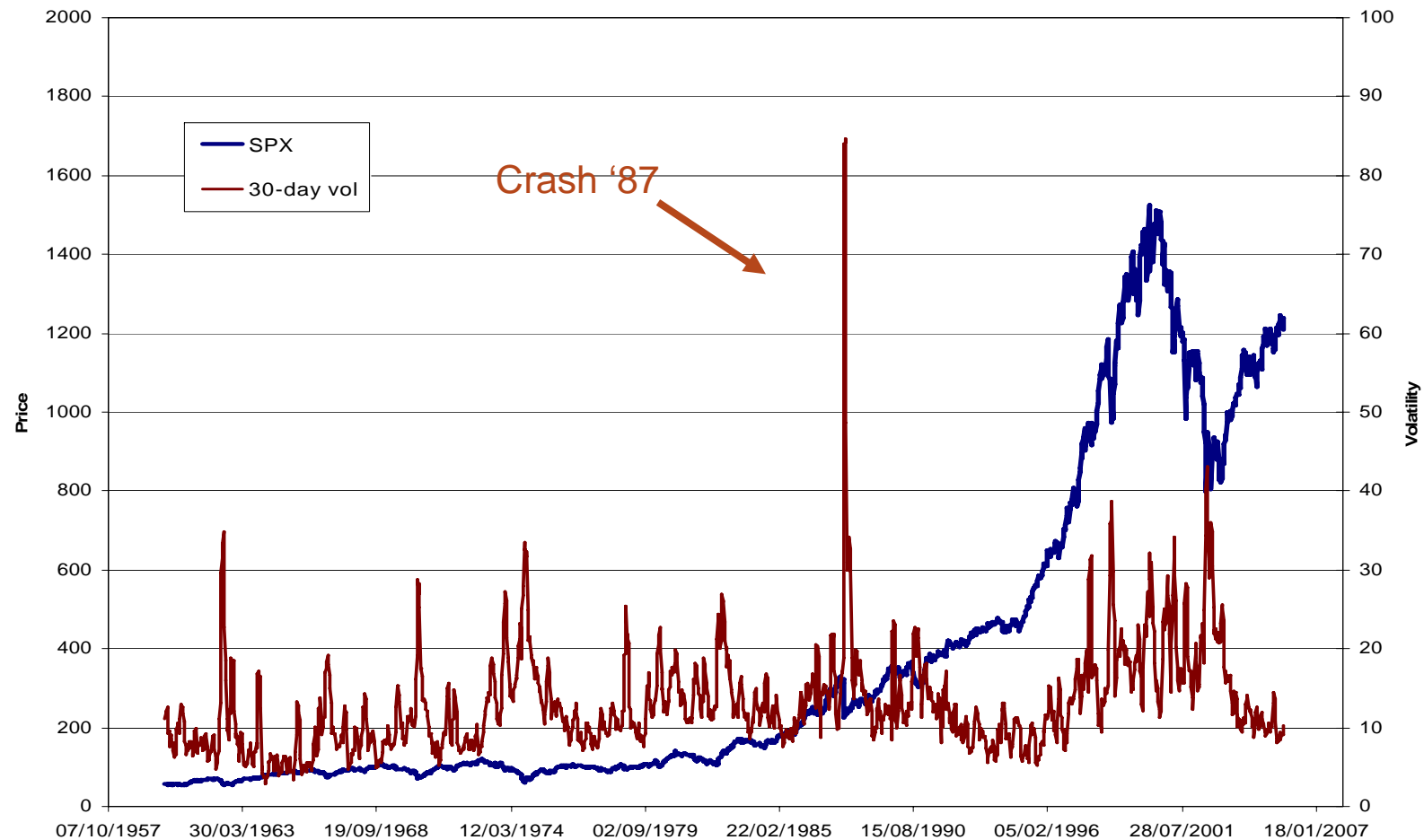
Mean-reversion level θ is a positive martingale.



Variance Curve Models

Why mean-reversion?

SPX Spot level and 30-day realized volatility





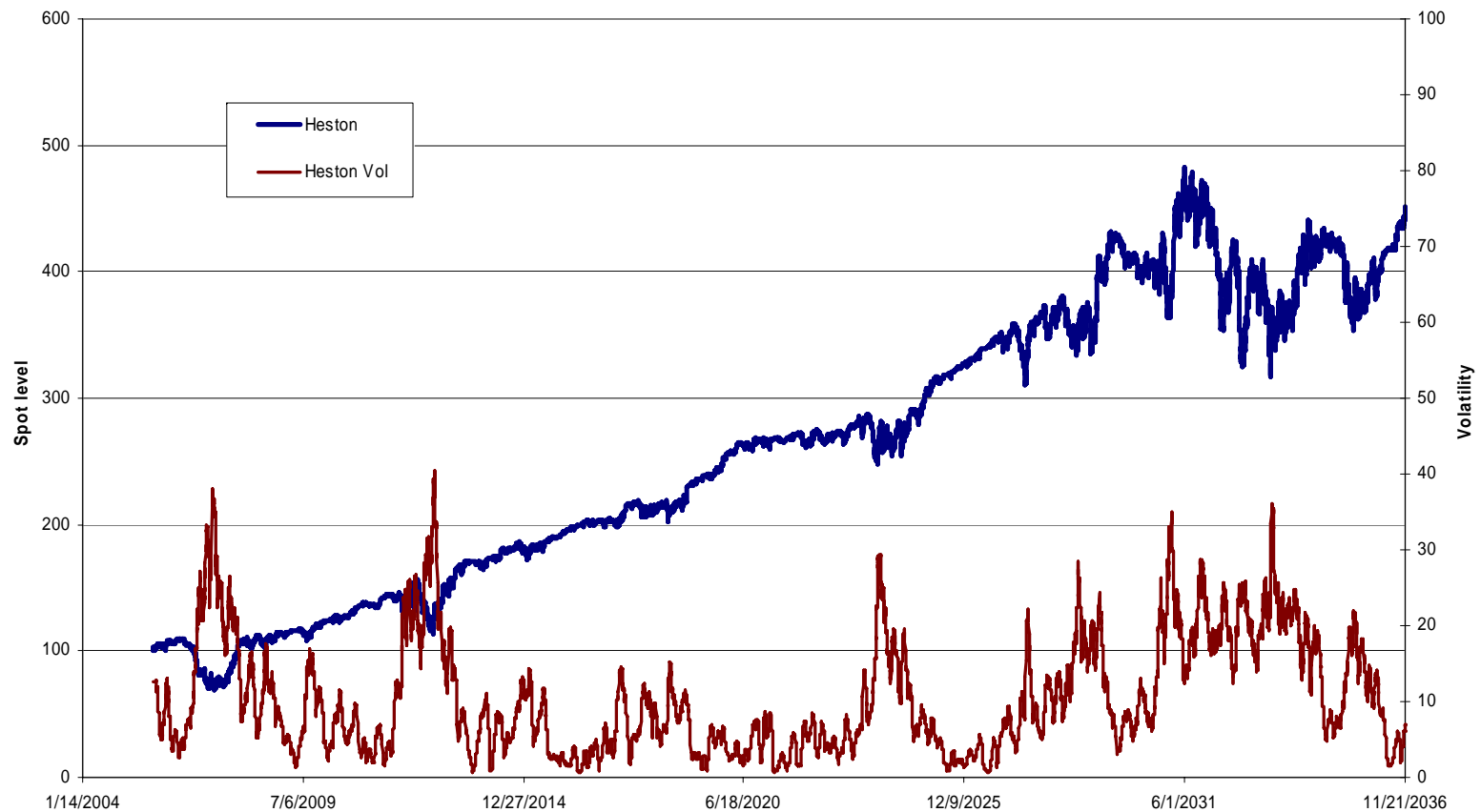
Variance Curve Models

Why mean-reversion?

Unconstrained Calibration

ShortVol	14.4%
LongVol	28.7%
RevSpeed	0.23
Correlation	-0.74
VolOfVol	26.3%

Heston path and 30-day realized volatility



Small print: this is quite an ideal sample path. Heston is not always to convincing.



Variance Curve Models

Variance Curve Functionals

■ Proposition

The observation that mean-reversion speeds must be constant holds for all polynomial-exponential functionals, i.e. if

$$G(z_1, \dots, z_n, z_{n+1}, \dots, z_m; x) = \sum_{i=1}^n p_i(z; x) e^{-z_i x}$$

(where $(p_i)_i$ are polynomials), then the first n components must be constant (cf. Filipovic 2001 for interest rates).

■ A similarly restrictive result can be shown for functionals of the form

$$G(z_1, \dots, z_n, z_{n+1}, \dots, z_m; x) = \exp \left\{ \sum_{i=1}^n p_i(z; x) e^{-z_i x} \right\}$$

- The parameters in the exponent come in pairs, where one is twice as large as the other (again Filipovic 2001).



Variance Curve Models

Variance Curve Functionals – Example linear mean-reversion

- Another example of the polynomial-exponential class is

$$G(z; x) = z_3 + (z_1 - z_2)e^{-\kappa x} + (z_2 - z_3)\frac{\kappa}{\kappa - c}\left(e^{-cx} - e^{-\kappa x}\right)$$

- A consistent factor model for this G must have the form

$$\begin{aligned}dZ_t^1 &= \kappa(Z_t^2 - Z_t^1)dt + \sigma_1(Z_t)dW_t \\dZ_t^2 &= c(Z_t^3 - Z_t^2)dt + \sigma_2(Z_t)dW_t \\dZ_t^3 &= \sigma_3(Z_t)dW_t\end{aligned}$$

which we call “double mean-reverting”.

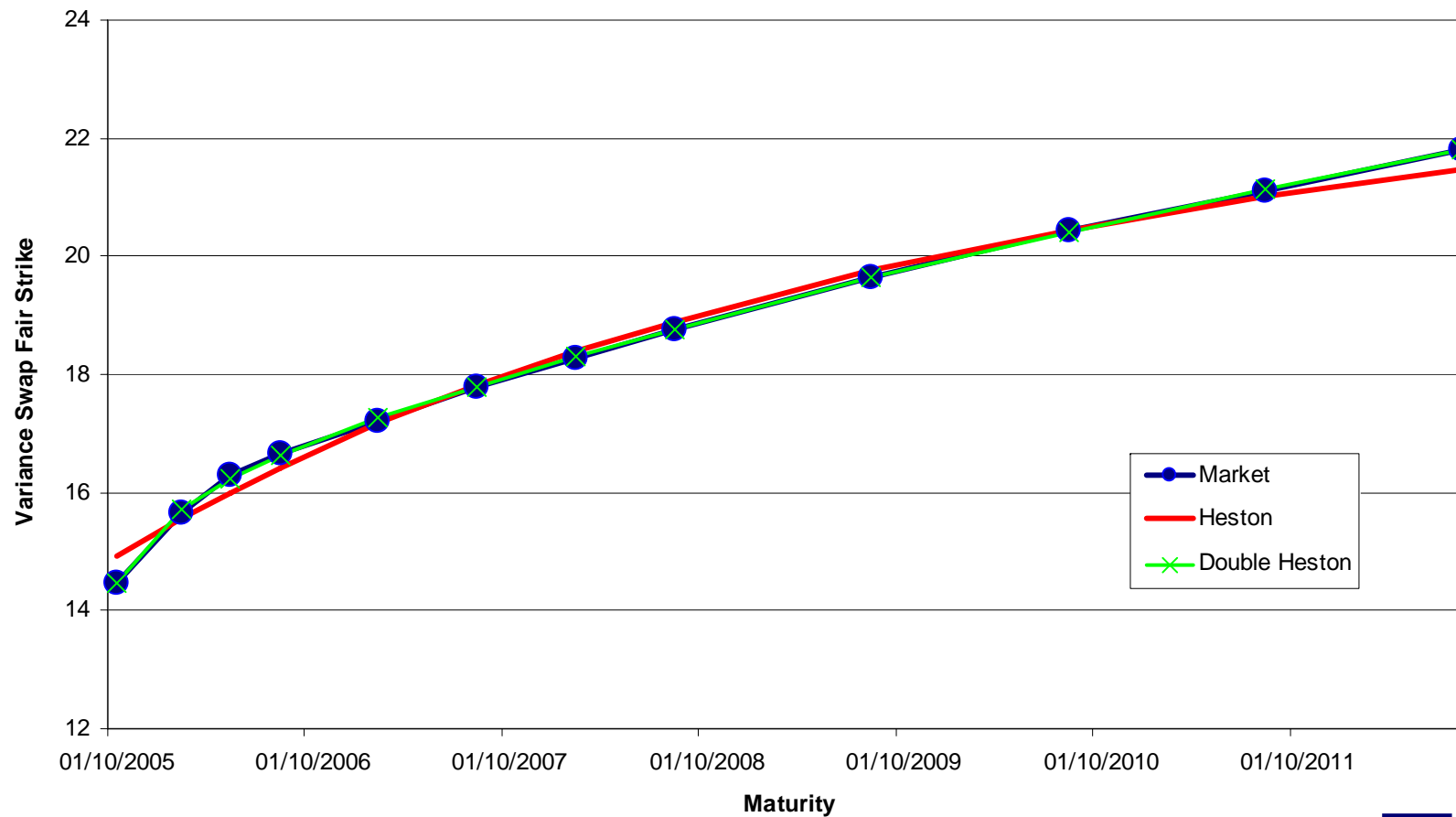
- Quite a good fit for most indices (at least during the course of the last year).



Variance Curve Models

Variance Curve Functionals – Example linear mean-reversion

Variance Swap Term Structure .SPX 10/12/2005





Hedging

Using Variance Curve Models to hedge Options on Variance



Hedging

How to hedge with variance curve models

- We are back in our initial classical setting, i.e. we have decided to use a consistent variance curve model,

$$\hat{v}_t(x) = G(Z_t; x)$$

$$dZ_t = \mu(Z_t)dt + \sum_{j=1, \dots, d} \sigma^j(Z_t) dW_t^j$$

$$\zeta_t := \hat{v}_t(0)$$

- We want to price and hedge an “option on realized variance” with (bounded) European payoff h ,

$$h\left(\int_0^T \zeta_s ds\right)$$



Hedging

How to hedge with variance curve models

- A general statement:

- If the vector of liquid instruments $S=(S^1, \dots, S^K)$ in a Market is a diffusion, and if the operator

$$f(t, x) := \mathbb{E}[F(X_t) | X_0 = x]$$

maps bounded C^∞ functions with bounded derivatives to C^1 functions, then the market of the “relevant payoffs”, i.e. those depending on S , is complete.

(Details and more general cases are work in progress).



Hedging

How to hedge with variance curve models

- Hence our candidate price process for h is the martingale:

$$H_t := \mathbb{E} \left[h \left(\int_0^T \zeta_s ds \right) \mid \mathbb{F}_t \right]$$

- Due to the Markov-property of Z , we have

$$H_t = C_t(Z_t, V_t(t)) := \mathbb{E} \left[h \left(\int_0^T \zeta_s ds \right) \mid Z_t; V_t(t) \right]$$

Running realized variance



- The idea is now to express Z in terms of a finite number of variance swaps.



Hedging

How to hedge with variance curve models

- Define the *variance swap price* function

$$\bar{G}(z; x) := \int_0^x G(z; y) dy$$

and assume that there exist constant $0 < \varepsilon < \tau_1 < \dots < \tau_m$ such that

$$\bar{G}_{t_1, \dots, t_m}(z) := (\bar{G}(z; t_1), \dots, \bar{G}(z; t_m))$$

is invertible for all $t_k := \tau_k - \tau$ for $0 \leq \tau \leq \varepsilon$.

- This then allows to recover Z in any interval $[a, b]$ by

$$Z_t = \bar{G}_{T_1-t, \dots, T_m-t}^{-1} \left(\underset{\substack{\uparrow \\ \text{Variance swap}}}{V_t(T_1)} - V_t(t), \dots, \underset{\substack{\uparrow \\ \text{Running realized variance}}}{V_t(T_m)} - V_t(t) \right)$$

where $T_k := a + \tau_k$.

Variance swap

Running realized variance



Hedging

How to hedge with variance curve models

- Recall

$$H_t = C_t(Z_t, V_t(t)) := \mathbb{E} \left[h \left(\int_0^T \zeta_s ds \right) \mid Z_t; V_t(t) \right]$$

- To hedge this payoff in the interval $[a, b]$, we can write it due to our assumptions on G as

$$C_t(Z_t, V_t(t)) \equiv C_t(V_t(T_1), \dots, V_t(T_m), V_t(t))$$

such that (under the assumption that C is C^1)

$$dC_t(\dots) = \sum_{k=1}^m \partial_{V_k} C_t(\dots) dV_t(T_k)$$



Hedging

How to hedge with variance curve models

- This

$$dC_t(\dots) = \sum_{k=1}^m \partial_{V_k} C_t(\dots) dV_t(T_k)$$

is the desired hedge of h in terms of variance swaps.

- For options on variance, this is a “natural” hedge.
 - It can also be used for standard options (a delta-term for S will appear).
 - For forward started options, correlation (skew) risk should be taken into account.
- In practise, the above “VarSwapDelta” hedging ratios are computed via bumping of the variance swap price.



Variance Curves

Future



Variance Curves

Future

- Challenges ahead
 - Incorporation of stochastic interest rates and dividends (in particular long-term deals could exhibit strong exposure to stochastic interest rates).
 - Jumps both in the underlying and the variance process (witness S&P return graph earlier).
 - Correlation between the variance curves between different underlyings (our suspicion is that it is actually quite high).

- The dream, however, is to characterize the entire implied volatility surface or, equivalently, the “forward probability distribution” of S .



Thank you very much for your attention.

Working paper and at <http://www.math.tu-berlin.de/~buehler/>



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