Quantitative Research

Consistent Variance Curve Models

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Hans Buehler
http://www.math.tu-berlin.de/~buehler/
hans.buehler@db.com

Deutsche Bank

Realized Variance

Trading volatility

Consistent Variance Curve Models

Introduction

- We are given an equity index (S&P, EuroSTOXX,…).
- Listed options and complex OTC products are traded on S…
  - "Volatility" drives the price of such options.
  - Can we also *trade* "volatilty" ?

- Well, we can trade *realized variance*.
- It is typically computed over the business days $0=t_0<\ldots<t_m=T$ using the estimator

$$V^m(T) := \sum_{i=1}^{m} \left(\log S_{t_i} - \log S_{t_{i-1}}\right)^2$$

(up to scaling). But we will assume that it is actually given as

$$V^m(T) \approx \langle \log S \rangle_T$$

Consistent Variance Curve Models

Introduction

- We will assume that S is continuous, that it pays no dividends and that the interest rates are zero. Hence, we may write it as

$$\begin{aligned} S_t &= \exp\left(X_t - \tfrac{1}{2}\langle X \rangle_t\right) \\ dX_t &= \sqrt{z_t}\, dB_t \end{aligned}$$

Short variance

on a stochastic base $(W, \mathrm{P}, \mathrm{F})$.

– The one-dimensional Brownian motion B is adapted to the filtration F.

– The process z is a predictable, integrable and non-negative.

- Realized variance is then

$$\langle \log S \rangle_T = \int_0^T z_s\, ds$$

Realized Variance

Variance Swaps

- The simplest product on realized variance is a *variance swap*.
- A variance swap is just a forward on realized variance:
  - At maturity T it pays the realized variance occurred during the life of the contract.
  - Such contracts are today liquidly traded on most major indices.
    In particular, their price processes are martingales on $(W, \mathrm{P}, \mathrm{F})$.
  - Hence, the price $V_t(T)$ of a variance swap is just the expectation of the realized variance,

$$V_t(T) = \mathrm{E}\left[\int_0^T z_s \, ds \mid \mathrm{F}_t \right]$$

Realized Variance

Variance Swaps

- If European options are traded for all strikes, the price of a variance swap can in theory be computed in terms of European options using Neuberger's (1990) formula,

$$
\begin{aligned}
V_0(T) &= 2\,\mathrm{E}\left[-\int_0^T \sqrt{z_s}\,dB_s + \tfrac{1}{2}\int_0^T z_s\,ds\right] \\
&= 2\,\mathrm{E}\left[S_T - 1 - \log S_T\right] \\
&= 2\left\{\int_0^1 \frac{1}{K^2}\mathrm{Put}(T,K)\,dK + \int_1^\infty \frac{1}{K^2}\mathrm{Call}(T,K)\,dK\right\}
\end{aligned}
$$

  – This works only if option prices are available for all T.

- The formula probably contributes to the fact that variance swaps are now liquidly traded.

Realized Variance

Forward Variance

- By construction,

$$V_t(T) = \mathrm{E}\left[\int_0^T z_s ds \mid \mathcal{F}_t\right] = \int_0^T \mathrm{E}\left[z_s \mid \mathcal{F}_t\right] ds$$

it is clear that we can derive V in T and obtain the *forward variance* v given as

$$v_t(T) := \partial_T V_t(T) = \mathrm{E}\left[z_T \mid \mathcal{F}_t\right]$$

Observation time **Forward time**

– Note the similarity to the *forward rate* in interest rates.

– We therefore aim to apply the ideas from interest rate theory to the (forward) variance curves.

Realized Variance

Forward Variance

- In fact, as in interest rates, we now want to specify the curve v instead of S and z….

Variance Curve Models

Standard approach

Variance Curve Models

Classic approach

- First, we focus on the classical setup.
  Assume we have a driving n-dimensional extremal Brownian motion W on the space (Ω,P,F).

- Definition
  A family $v=(v(T))_{T\geq 0}$ is called a *Variance Curve Model* if

1. For each $T>0$, the process $v(T)=(v_t(T))_{t\in[0,T]}$ is a martingale;

$$dv_t(T) = b_t(T)dW_t \qquad b_t(T) \in L^{\text{loc}}$$

2. For each $T>0$, the initial variance swap prices are finite, i.e.

$$V_0(T) = \int_0^T v_0(s)ds < \infty$$

3. The curve $v_t(\cdot)$ is continuous (jointly measurable).

Variance Curve Models

Classic approach

- Properties
  - The price processes of variance swaps,

$$V_t(T) := \int_0^T v_t(s)ds$$

    are martingales.

  - The *short variance process*

$$z_t := v_t(t)$$

    is well defined, integrable and non-negative.

Variance Curve Models

Classic approach

- Properties

  Given a standard Brownian motion B on $(W,\mathrm{P},\mathrm{F})$, the process

  $$dX_t = \sqrt{z_t}\, dB_t$$

  is a square-integrable martingale, so the via B "associated" stock price

  $$S_t := \exp\left(X_t - \tfrac{1}{2} \langle X \rangle_t \right)$$

  is a local martingale.

  $\rightarrow$ B represents the correlation structure of S with v.

- Theorem

  For each variance curve model v and each Brownian motion B, the market

  $$\left(S; (V(T))_{T \geq 0} \right)$$

  is free of arbitrage.

Variance Curve Models

Classic approach – Musiela-Parametrization

- As in interest rates, it is more convenient to work with fixed time-to-maturities $x:=T-t$. Hence we define the *Musiela parameterization*

$$\hat{v}_t(x) := v_t(t+x)$$

- Properties
  - Assume that

$$\int_0^\infty \int_t^\infty \partial_T \boldsymbol{b}_t(T)^2 \, dT dt < \infty$$

Then,

$$d\hat{v}_t(x) := \partial_x \hat{v}_t(x) dt + \hat{\boldsymbol{b}}_t(x) dW_t$$

where

$$\hat{b}_t(x) := b_t(t+x)$$

Variance Curve Models

Classic approach – Exponential Musiela-Parametrization

- If v is represented as an exponential,

$$\hat{v}_t(x) := v_o(x)\exp(\hat{w}_t(x))$$

it allows to work with an initial term-structure fit v_0.

  – We obtain the condition

$$d\hat{w}_t(x) = \left(\partial_x \hat{w}_t(x) - \tfrac{1}{2}\hat{b}_t(x)^2\right)dt + \hat{b}_t(x)dW_t$$

but additional care must be taken to ensure that v is a true martingale.

Variance Curve Models

Variance Curve Functionals

- All this was too general. In practice, we cannot easily simulate a curve. Rather, we need some interpolation scheme, i.e. a finite-dimensional representation of the curve we are using.

- We want to write

$$\hat{v}_t(x) := G(Z_t; x)$$

for some suitable non-negative function G and an m-dimensional Markov-process Z.

Variance Curve Models

Variance Curve Functionals

- <u>Definition</u>

1. A non-negative C[2,2]-function $G:DxR^{+}$ ® R^{+} is called a *Variance Curve Functional* if

$$\int_0^T G(z;x)dx < \infty$$

for all T and $z \in D$.

2. We denote by $\Xi(D)$ the space of all Markov-diffusions Z which are unique solutions to an SDE

$$dZ_t = m(Z_t)dt + s\,(Z_t)dW_t$$

and which stay inside the domain D of G.

Variance Curve Models

Variance Curve Functionals

- <u>Remark</u>
  If we only want to make local statement and if D is a sub-manifold, then we know that Z stays locally in D if and only if

$$m(z) - \sum_{i=1}^{n} s^{i}(z) \cdot \partial_z s^{i}(z) \in T_z D$$

$$s^{i}(z) \in T_z D$$

Stratonovic drift correction

- <u>Definition</u>
  A *Consistent Parameter Process* for G is a Markov-process $Z \in \Xi$ such that

$$\hat{v}_t(x) := G(Z_t; x)$$

defines a variance curve model.

Variance Curve Models

Variance Curve Functionals

- <u>Theorem</u>
  This is the case if and only if,

$$\partial_x G(Z_t;x) = m(Z_t)\partial_z G(Z_t;x) + \frac{1}{2} s^2(Z_t)\partial_{zz}^2 G(Z_t;x)$$

  P$x\,l$-a.s.

  – *Proof:*

    - If G and Z are consistent, $G(Z_t;T\text{-}t)$ is a martingale. The result follows from Ito.
    - If the above equation holds, then $v(T) := G(Z_t;T\text{-}\cdot)$ is a local martingale, again by Ito's formula. Indeed, it is a true martingale because $v_t(T) = \mathrm{E}[\, v_T(T) \mid F_t \,] = \mathrm{E}[G(Z_t;0 \mid F_t\,]$.

- Work in progress: Given G, when exists Z?
  (suspicion: G might have to be quasi-exponential)

Variance Curve Models

Term-structure approach

- Hence, model the curves as elements of an Hilbert space.
- Very similar to Bjoerk/Christensen (1999), Filipovic (2000), Filipovic/Teichmann (2004).
  - Clarify Hilbert space.
  - Write the curve as a Markov SDE in this space.
  - Show it is a variance curve model (i.e., variance swaps are martingales).
  - Apply Filipovic/Teichmann results...

Variance Curve Models

Term-structure approach

- For example, define as in Filipovic (2000) a Hilbert space H

$$\|h\|_H^2 := |h(0)|^2 + \int_0^\infty |h'(s)|^2 e^{as} ds \qquad (a > 0)$$

$$H := \left\{ h(x) = \int_0^x h'(s) ds \;\middle|\; \|h\|_H < \infty \right\}$$

  – Then, $h \in H$ is continuous, ∂_x is the generator of the semi-group of right-shifts and for all x, there exists a constant $u(x)$ such that

$$\left\| \int_0^x h(s) ds \right\|_H \leq x u(x)$$

Variance Curve Models

Term-structure approach

- The variance curve is Markov and satisfies

$$d\hat{v}_t := \partial_x \hat{v}_t dt + \hat{b}(\hat{v}_t)dW_t = \partial_x \hat{v}_t dt + \sum_{i=1.}^{n} \hat{b}^i(\hat{v}_t)dW_t^i$$

$$\hat{v}_0 \in M$$

where W could also be infinite-dimensional.

- In that case, the variance swaps

$$V_t(T) := V_t(t) + \int_0^{T-t} \hat{v}_t(x)dx$$

are martingales because of $\left\|\int_0^x h(s)ds\right\|_H \leq xu(x)$, hence we have a variance curve model.

Variance Curve Models

Term-structure approach

- Theorem (Filipovic/Teichmann 2004)
  A sub-manifold M is locally invariant iff

  1) $M \subset \mathrm{dom}(\partial x)$

  2) $b^0(\hat{v}) := \partial_x \hat{v} - \sum_{i=1,..} (b^i)'(\hat{v}) \cdot b^i(\hat{v})$ locally Lipschitz

  3) $\hat{b}^i(m) \in T_m M \quad i = 0,1,\ldots$

  Frechet-Derivative

- Existence
  Using results from Filipovic/Teichmann (2002), we then now that if the Lie-algebra implied by the Stratonovic-drift and the volatility coefficients is finite-dimensional for all curves from some "thin" set, then there exists a finite-dimensional representation.

  $\rightarrow$ What about possible shapes of M?

Back to reality: A Simple Example

Linear mean-reversion

Variance Curve Models

Variance Curve Functionals – Example linear mean-reversion

- Example
  The classic Heston (1993) model is given by the following SDE for the short variance:

LongVar

$$dz_t = k(q - z_t)dt + x\sqrt{z_t}\,dW_t^1$$

ReversionSpeed

VolOfVol

  – The forward variance is therefore

$$\hat{v}_t(x) = q + (z_t - q)e^{-kx}$$

  – Question: What dynamics can the parameter $z=(z_1, z_2, z_3)$ have for the curve functional

$$G(z;x) = z_2 + (z_1 - z_2)e^{-z_3 x}$$

with $(z_1,z_2,z_3>0)$?

Variance Curve Models

Variance Curve Functionals – Example linear mean-reversion

- The SDE coefficients m and s have to satisfy

$$\partial_x G(z;x) = m(z)\partial_z G(z;x) + \frac{1}{2}s^2(z)\partial^2_{zz} G(z;x)$$

$$G(z;x) := z_2 + (z_1 - z_2)e^{-z_3 x}$$

1. First, we see that
$$\partial^2_{z_3 z_3} G(z;x) = (z_1 - z_2)x^2 e^{-z_3 x}$$
Since no term x^2e^x appears on the left hand side, we must have $s_3=0$.
2. The same line of thought using
$$\partial_{z_3} G(z,x) = -(z_1 - z_2)xe^{-z_3 x}$$
shows that we also have $m_3=0$.

Hence, the mean-reversion speed cannot be stochastic.

Variance Curve Models

Variance Curve Functionals – Example linear mean-reversion

- For the other two parameters, we find that while s is unconstrained,

$$m_2(z) = 0$$

$$m_1(z) = z_3(z_2 - z_1)$$

In other words: The only consistent processes for this choice of G are of the type

Linear mean-reversion drift

$$dz_t = k(q_t - z_t)dt + s_1(z_t, q_t)dW_t$$

$$dq_t = s_2(z_t, q_t)dW_t$$

VolOfVol can freely be chosen as long as z remains non-negative.

Mean-reversion level q is a positive martingale.

Variance Curve Models

Variance Curve Functionals – Example linear mean-reversion

- A convenient choice is

$$\begin{aligned} dz_t &= k(q_t - z_t)dt + xz_t^a d(rW_t^1 + \sqrt{1-r^2}W_t^2) \\ dq_t &= nq_t dW_t^1 \end{aligned}$$

for $\alpha \in (0.5, 1]$

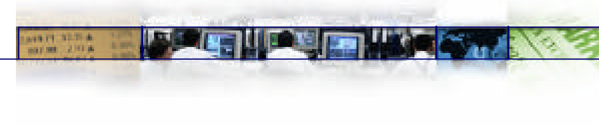
- <u>Proposition</u>
The same holds for all polynomial-exponential functionals, i.e. if

$$G(z_1, \ldots, z_m, z_{m+1}, \ldots, z_d; x) = \sum_{i=1}^{m} p_i(z; x) e^{-z_i x}$$

(where $(p_i)_i$ are polynomials), then the first m components must be constant (cf. Filipovic 2001 for interest rates).

Hedging in Practice

A crash course: Using Variance Curve Models to hedge Options on Variance



Hedging

How to hedge with variance curve models

- We are back in our initial classical setting, i.e. we have decided to use a consistent variance curve model,

$$\hat{v}_t(x) = G(Z_t; x) \quad Z_t \in D \subset R^m$$

$$dZ_t = m(Z_t)dt + s(Z_t)dW_t$$

$$z_t := \hat{v}_t(0)$$

- Now an investor wants to buy an "option on realized variance" with European payoff H. Its price is given as

$$C_t(T) := \mathrm{E}\left[H\left(\int_0^T z_s ds \right) \middle| F_t \right]$$

Hedging

How to hedge with variance curve models

- Hedging
  - Due to the Markov-property of Z, we have

$$C_t(T; Z_t; \int_0^t z_s ds) = \mathrm{E}\left[H\left(\int_0^T z_s ds \right) \middle| Z_t; \int_0^t z_s ds \right]$$

assume now that for every interval $[a,b] \subset [0,T]$, there are $\kappa \le m$ variance swaps with maturities $b < T_1 < \cdots < T_k$ such that

$$Z_t = \left(G^{-1}(V_t(T_i); T_i - t), i = 1 \ldots k \right)$$

for $t \in [a,b)$.

  - Hence, we can write

$$\tilde{C}_t(T; (V_t(T_i))_{i=1,\ldots,m}; \int_0^t z_s ds) := C_t(T; G^{-1}(V_t; \cdots); \int_0^t z_s ds)$$

Hedging

How to hedge with variance curve models

- As a result,

$$d\tilde{C}_t = \sum_{i=1}^{m} \partial_{V_i} \tilde{C} \, dV_t(T_i)$$

hence we obtain a hedge in terms of our variance swaps.

- For options on variance, this is a “natural” hedge.
- It can also be used for standard options (a delta-term will appear).
- For Cliquets, correlation (skew) risk should be taken into account.

Thank you very much for your attention.

Working paper at http://www.math.tu-berlin.de/~buehler

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