


## Consistent Variance Curve Models

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1,100.02	30.07 ▲	1,130.00
2,649.71	33.35 ▲	2,683.06
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1,100.02	30.07 ▲	1,130.00





# Consistent Variance Curve Models

## Outline

### ■ Introduction

- Options on realized variance

### ■ Variance Curve Models

- General theory
- Finite-dimensionally parameterized curves
- Variance Curves in a Hilbert space

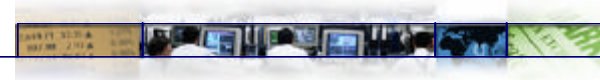
### ■ Examples

### ■ Hedging



# Realized Variance

Trading volatility



# Consistent Variance Curve Models

## Introduction

- We are given an equity index (S&P, EuroSTOXX,...).
- Listed options and complex OTC products are traded on  $S$ .
  - “Volatility” drives the price of such options.
  - Can we also *trade* “volatility” ?
- Well, we can trade *realized variance*.
- It is typically computed over the business days  $0=t_0 < \dots < t_N=T$  using the estimator

$$V^N(T) := \sum_{i=1}^N \left( \log S_{t_i} - \log S_{t_{i-1}} \right)^2$$

(up to scaling). But we will assume that it is actually given as

$$V^N(T) \approx \langle \log S \rangle_T$$



# Consistent Variance Curve Models

## Introduction

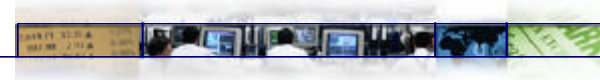
- We will assume that  $S$  is continuous, that it pays no dividends and that the interest rates are zero. Hence, we may write it as

$$S_t = \exp\left(X_t - \frac{1}{2}\langle X \rangle_t\right)$$
$$dX_t = \sqrt{\mathbf{z}_t} dB_t$$

on a stochastic base  $(W, \mathbb{P}, \mathbb{F})$

- The one-dimensional Brownian motion  $B$  is adapted to the filtration  $\mathbb{F}$ .
  - The *short variance* process  $\mathbf{z}$  is a predictable, integrable and non-negative.
- Realized variance is then the non-negative quantity

$$\langle \log S \rangle_T = \int_0^T \mathbf{z}_s ds$$



# Realized Variance

## Variance Swaps

- The simplest product on realized variance is a *variance swap*.
- A variance swap is just a forward on realized variance:
  - At maturity  $T$  it pays the realized variance occurred during the life of the contract (usually in exchange for a previously agreed fixed strike  $K$ ).
  - Such contracts are today liquidly traded on most major indices. In particular, their price processes are martingales under each equivalent martingale measure on  $(W, P, F)$ .
    - Let us assume that  $P$  is a martingale measure.
  - Then, the price  $V_t(T)$  of a variance swap is just the expectation of the realized variance,

$$V_t(T) = E \left[ \int_0^T \mathbf{z}_s ds \mid F_t \right]$$



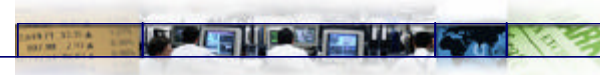
# Realized Variance

## Variance Swaps

- If European options are traded for all strikes, the price of a variance swap can in theory be computed in terms of European options using Neuberger's (1990) formula,

$$\begin{aligned} V_0(T) &= 2 \mathbb{E} \left[ - \int_0^T \sqrt{\mathbf{z}_s} dB_s + \frac{1}{2} \int_0^T \mathbf{z}_s ds \right] \\ &= 2 \mathbb{E} \left[ S_T - 1 - \log S_T \right] \\ &= 2 \left\{ \int_0^1 \frac{1}{K^2} \text{Put}(T, K) dK + \int_1^\infty \frac{1}{K^2} \text{Call}(T, K) dK \right\} \end{aligned}$$

- This works only if option prices are available for all  $T$ .
- The formula probably contributes to the fact that variance swaps are now liquidly traded.
  - An excellent reference is Demeterfi et al (1999).

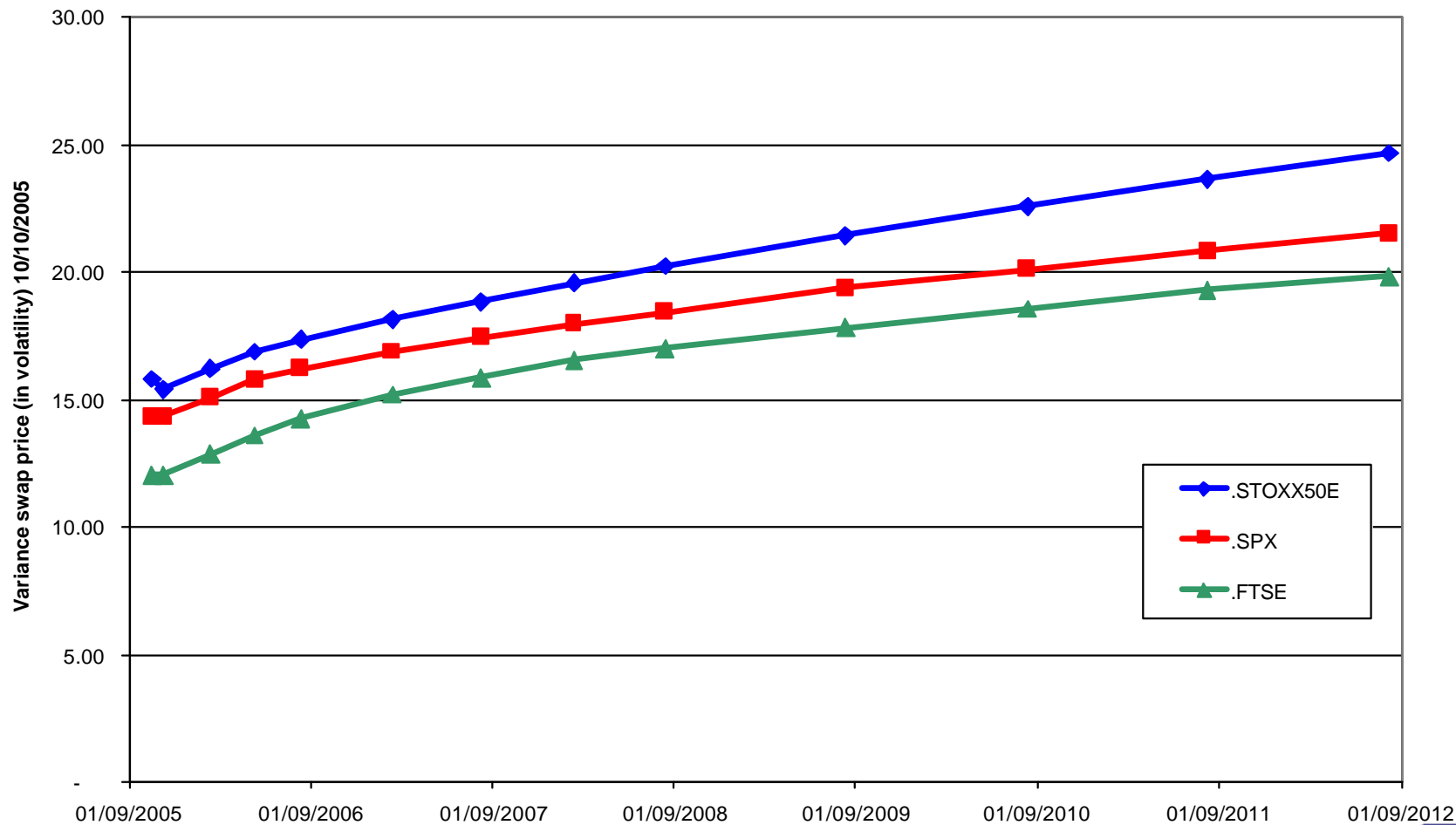


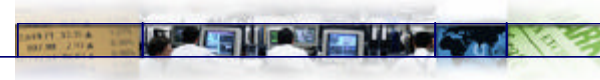
# Realized Variance

## Variance Swaps

Prices are quoted in "volatility"

$$\sqrt{\frac{1}{T} \int_0^T \mathbf{z}_s ds}$$





# Realized Variance

## Beyond Variance Swaps

- Since variance swaps are liquidly traded, there is no need to price them.
- But what about more complex products:

– *Calls* on realized variance

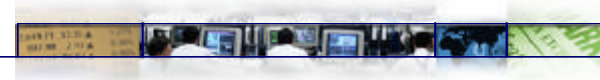
$$\left( \int_0^T \mathbf{z}_s ds - K^2 \right)^+$$

– *Volatility swaps*

$$\sqrt{\int_0^T \mathbf{z}_s ds} - K$$

– But also *forward started options*

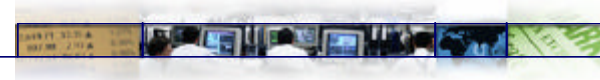
$$\left( \frac{S_{T_2}}{S_{T_1}} - k \right)^+ = \left( \exp \left( \int_{T_1}^{T_2} \sqrt{\mathbf{z}_s} dB_s - \frac{1}{2} \int_{T_1}^{T_2} \mathbf{z}_s ds \right) - k \right)^+$$



# Realized Variance

## Beyond Variance Swaps

- The idea is to use the variance swaps themselves as underlying reference instruments.
  - A variance swap is a natural hedging instrument for such payoffs.
  - We can then attempt to hedge options on realized variance by delta-hedging with variance swaps.
  
- Mathematically, the term-structure of variance swaps reminds on the term-structure of discount bounds in interest rate models
  - *A term-structure problem is looming!*



# Realized Variance

## Forward Variance

- Variance swap prices are increasing with maturity  $T$ .
  - Their price at a later time  $t$  also depends on the past realized variance.
- To alleviate these unpleasant properties, note that

$$V_t(T) = \mathbb{E} \left[ \int_0^T \mathbf{z}_s ds \mid \mathbb{F}_t \right] = \int_0^T \mathbb{E}[\mathbf{z}_s \mid \mathbb{F}_t] ds$$

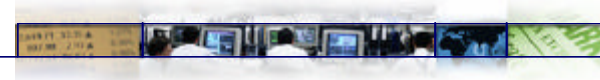
can be differentiated in  $T$  to define the *forward variance*

$$v_t(T) := \partial_T V_t(T) = \mathbb{E}[\mathbf{z}_T \mid \mathbb{F}_t]$$

Observation time

Maturity

- Note the similarity to the *forward rate* in interest rate theory.

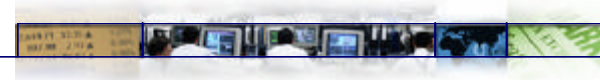


# Realized Variance

## Modeling forward Variance

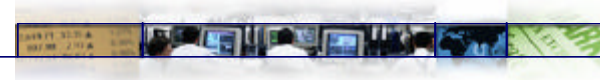
### ■ Idea

- Instead of starting with  $S$ , let us first specify the non-negative curve  $v$ .
- Then, construct a (local) martingale  $S$  which has the correct quadratic variation.
- The model shall also yield hedging ratios in terms of variance swaps.
- The main difference is that forward variance can be zero for good economic reasons (holidays, suspended trading).
  - Short variance also becomes zero for standard stochastic volatility models such as Heston's.



# Variance Curve Models

A structural approach



# Variance Curve Models

## Classic approach

- First, we focus on the classical setup.  
Assume we have a driving  $d$ -dimensional extremal Brownian motion  $W$  on the space  $(\Omega, \mathcal{P}, \mathcal{F})$ .

- Definition

A family  $v = (v(T))_{T \geq 0}$  is called a *Variance Curve Model* if

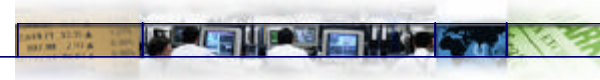
1. For each  $T > 0$ , the process  $v(T) = (v_t(T))_{t \in \hat{I}[0, T]}$  is a non-negative martingale:

$$dv_t(T) = \sum_{j=1, \dots, d} \mathbf{b}_t^j(T) dW_t^j \quad \mathbf{b}^j(T) \in L^{\text{loc}}$$

2. For each  $T > 0$ , the initial variance swap prices are finite, i.e.

$$V_0(T) = \int_0^T v_0(s) ds < \infty$$

3. The curve  $v_t(t)$  is left-continuous.



# Variance Curve Models

Classic approach

## ■ Properties

- The price processes of variance swaps,

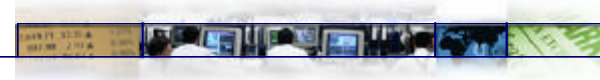
$$V_t(T) := \int_0^T v_t(s) ds$$

are martingales.

- The *short variance process*

$$Z_t := v_t(t)$$

is well defined, integrable and non-negative.



# Variance Curve Models

## Classic approach

### ■ Properties

Given any standard Brownian motion  $B$  on  $(W, P, F)$ , the process

$$dX_t = \sqrt{\mathbf{z}_t} dB_t$$

is a square-integrable martingale, so the via  $B$  *associated stock price*

$$S_t := \exp\left(X_t - \frac{1}{2} \langle X \rangle_t\right)$$

is a local martingale.

- $B$  represents the *correlation structure* of  $S$  with  $v$ .

### ■ Theorem

For each variance curve model  $v$  and each Brownian motion  $B$ , the market

$$\left(S; (V(T))_{T \geq 0}\right)$$

is free of arbitrage.



# Variance Curve Models

Classic approach – Musiela-Parametrization

- As in interest rates, it is more convenient to work with fixed time-to-maturities  $x:=T-t$ . Hence we define the *Musiela parameterization*

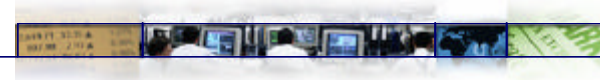
$$\hat{v}_t(x) := v_t(t+x)$$

- Starting in Musiela-parametrization

- Assume that  $\sum_{j=1,\dots,d} \int_0^\infty \int_t^\infty \partial_T \mathbf{b}_t(T)^2 dT dt < \infty$   
Then,

$$d\hat{v}_t(x) := \partial_x \hat{v}_t(x) dt + \sum_{j=1,\dots,d} \hat{\mathbf{b}}_t^j(x) dW_t^j$$

defines a variance curve model in Musiela-parametrization.



# Variance Curve Models

Classic approach – Fitting the market

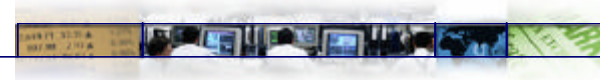
- If  $\nu$  is represented as an exponential,

$$\hat{\nu}_t(x) := \exp(\hat{w}_t(x))$$

it allows us to fit the model easily to an observed market forward variance curve  $u_0$  by setting  $w_0 := \log u_0$ .

- This construction does not allow  $\nu$  to become zero.
- A more convenient approach is to use an existing curve  $\nu^{\text{base}}$  and set

$$\hat{\nu}_t(x) := \frac{u_0(t+x)}{\hat{\nu}_0^{\text{base}}(t+x)} \hat{\nu}_t^{\text{base}}(x)$$



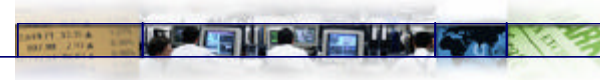
# Variance Curve Models

## Variance Curve Functionals

- Problems with a specification with general integrands  $\mathbf{b}(T)$ :
  - It is complicated to check whether  $\nu$  remains non-negative.
  - In practice, it is not clear how to handle such integrands computationally.
- Hence, we want to write

$$\hat{\nu}_t(x) := G(Z_t; x)$$

for some suitable non-negative function  $G$  and an  $m$ -dimensional Markov-process  $Z$ .



# Variance Curve Models

## Variance Curve Functionals

### ■ Definition

1. A non-negative  $C^{2,2}$ -function  $G: D \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is called a *Variance Curve Functional* if

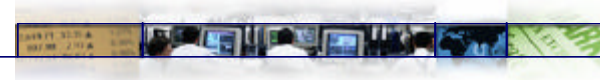
$$\int_0^T G(z; x) dx < \infty$$

for all  $T$  and  $z \in D$  where  $D$  is an open set in  $\mathbb{R}_{\geq 0}^m$ .

2. We denote by  $\Xi$  the set of all  $C=(\mathbf{m}, \mathbf{s})$  for which the SDE

$$dZ_t = \mathbf{m}(Z_t)dt + \sum_{j=1, \dots, d} \mathbf{s}^j(Z_t) dW_t^j$$

starting at any point  $Z_0 \in D$  has a unique solution  $Z$  which stays in  $D$ .



# Variance Curve Models

## Variance Curve Functionals

### ■ Definition

We call  $C=(\mathbf{m},\mathbf{s})\in\Xi$  a *consistent factor model* for  $G$  if for any  $Z_0\hat{\mathbf{I}}D$ ,

$$\hat{v}_t(x) := G(Z_t; x)$$

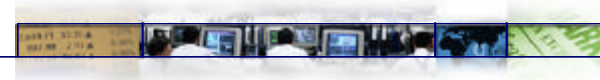
defines a variance curve model.

### ■ Theorem

This is the case if and only if  $Z$  stays in  $D$  and if

$$\partial_x G(z; x) = \mathbf{m}(z)\partial_z G(z; x) + \frac{1}{2}\mathbf{s}^2(z)\partial_{zz}^2 G(z; x)$$

holds such that  $G(Z_t; T-t)$  is a true martingale.



# Variance Curve Models

## Variance Curve Functionals

### ■ Remarks

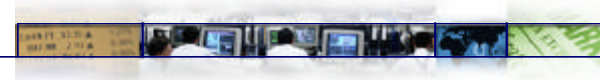
- For each consistent pair  $(G, C)$ , we obviously have

$$G(z; x) \equiv \mathbb{E}[G(Z_x; 0) \mid Z_0 = z]$$

- But our interest was asked the reverse question:  
given  $G$ , find a consistent  $C=(\mathbf{m}, \mathbf{s}) \in \Xi$ .

### ■ The next logical step is to model the curve $\nu$ as a process with values in a Hilbert space $H$ .

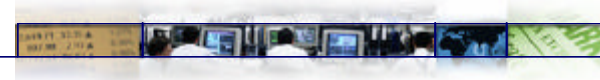
- We follow the path laid by Bjoerk/Christensen (1999), Filipovic (2000), Filipovic/Teichmann (2004) and Teichmann (2005).



# Variance Curve Models

## Term-structure approach

- The main difference between variance curve and forward curves is that the curves  $\nu$  must remain non-negative (but *can* become zero).
  - The problem is that the “non-negative cone” is a very small set. Indeed it has no interior points.
  - However, if  $G(D)$  is a sub-manifold with boundary of  $H$ , then it is sufficient to check whether  $\nu$  stays in  $G(D)$ .  
In this case we say  $G(D)$  is *locally invariant* for  $\nu$ .
  - If  $G$  is moreover invertible, we can also directly construct a (locally) consistent factor model  $C=(\mathbf{m}\mathbf{s})$  for  $G$ .



# Variance Curve Models

## Term-structure approach

- Assume that the variance curve  $v$  is given as a solution in  $H$  to

$$d\hat{v}_t = \partial_x \hat{v}_t dt + \sum_{j=1, \dots, d} \hat{\mathbf{b}}^j(\hat{v}_t) dW_t^j$$

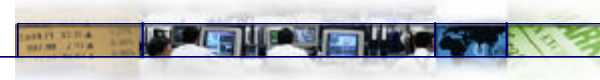
where the coefficients  $\mathbf{b}$  are locally Lipschitz vector fields.

- The Stratonovic-drift for  $v$  is as usual

$$\mathbf{b}^0(\hat{v}) := \partial_x \hat{v} - \sum_{j=1, \dots, d} D\mathbf{b}^j(\hat{v}) \cdot \mathbf{b}^j(\hat{v})$$

such that

$$d\hat{v}_t = \hat{\mathbf{b}}^0(\hat{v}_t) dt + \sum_{j=1, \dots, d} \hat{\mathbf{b}}^j(\hat{v}_t) \circ dW_t^j$$



# Variance Curve Models

## Term-structure approach

### ■ Theorem (Filipovic/Teichmann 2004)

The sub-manifold  $G(D)$  is locally invariant for  $\nu$  iff

1. We have  $G(D) \subset \text{dom}(\partial x)$ ,
2. In the interior of  $G(D)$ , we have

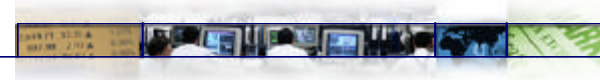
$$\hat{\mathbf{b}}^j(\hat{\nu}) \in T_{\hat{\nu}}G(D) \quad j = 0, \dots, d$$

3. On the boundary  $\partial G(D)$ ,

$$\hat{\mathbf{b}}^0(\hat{\nu}) \in (T_{\hat{\nu}}G(D))_{\geq 0}$$

$$\hat{\mathbf{b}}^j(\hat{\nu}) \in T_{\hat{\nu}}\partial G(D) \quad j = 1, \dots, d$$

holds.



# Variance Curve Models

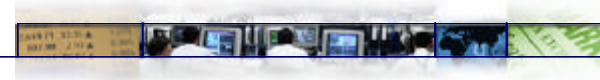
Term-structure approach

- If we can invert  $G$ , then  $C=(\mathbf{m},\mathbf{s})$  with

$$\mathbf{s}^j(z) := \partial_z G^{-1}(\hat{\mathbf{b}}^j(G(z)))$$

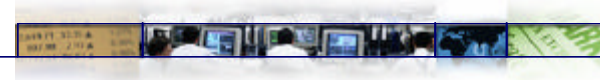
$$\mathbf{m}(z) := \partial_z G^{-1}(\hat{\mathbf{b}}^0(G(z))) + \sum_{j=1,\dots,d} (\partial_z \mathbf{s}^j)(z) \mathbf{s}^j(z)$$

is (locally) a consistent factor model for  $G$ .



# Back to reality: A Simple Example

Linear mean-reversion



# Variance Curve Models

## Variance Curve Functionals – Linear mean-reversion

### ■ Example

A very basic example is the “linearly mean-reverting” functional:

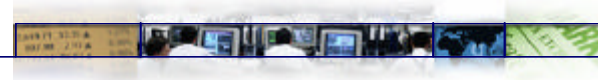
$$G(z; x) = z_2 + (z_1 - z_2)e^{-z_3 x}$$

„Long variance“

„Short variance“

„Speed of mean-reversion“

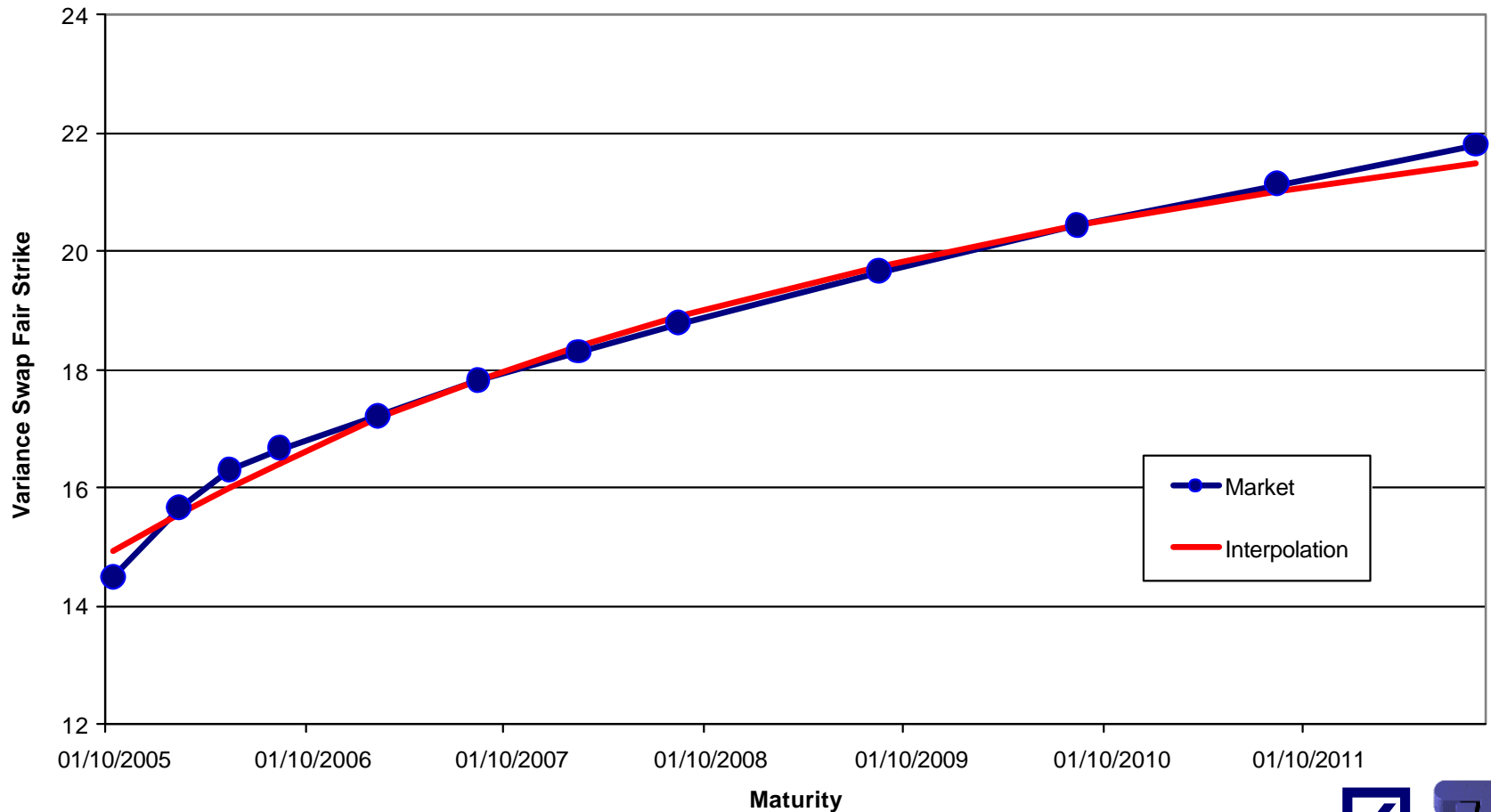
for  $z_1 \geq 0$  and  $z_2, z_3 > 0$ .

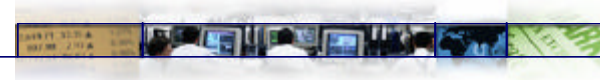


# Variance Curve Models

## Variance Curve Functionals – Linear mean-reversion

Variance Swap Term Structure .SPX 10/12/2005





# Variance Curve Models

## Variance Curve Functionals – Linear mean-reversion

- Question: What dynamics can a consistent process  $Z=(Z_1, Z_2, Z_3)$  have?
- The coefficients  $\mathbf{m}$  and  $\mathbf{s}$  have to satisfy

$$\partial_x G(z; x) = \mathbf{m}(z) \partial_z G(z; x) + \frac{1}{2} \mathbf{s}^2(z) \partial_{zz}^2 G(z; x)$$

1. First, we see that

$$\partial_{z_3 z_3}^2 G(z; x) = (z_1 - z_2) x^2 e^{-z_3 x}$$

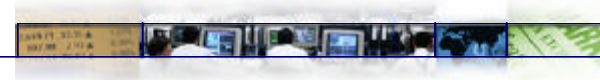
Since no term  $x^2 e^x$  appears on the left hand side, we must have  $\mathbf{s}_3=0$ .

2. The same line of thought applied to

$$\partial_{z_3} G(z, x) = -(z_1 - z_2) x e^{-z_3 x}$$

shows that we also have  $\mathbf{m}_3=0$ .

**Hence, the speed of mean-reversion cannot be stochastic.**



# Variance Curve Models

## Variance Curve Functionals – Linear mean-reversion

- For the other two parameters, we find that while  $s$  is unconstrained,

$$m_2(z) = 0$$

$$m_1(z) = z_3(z_2 - z_1)$$

In other words: The only consistent processes for this choice of  $G$  are of Heston-type

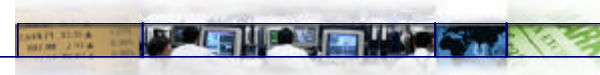
$$dz_t = k(q_t - z_t)dt + s_1(z_t, q_t)dW_t$$

$$dq_t = s_2(z_t, q_t)dW_t$$

Linear mean-reversion drift

VolOfVol can freely be chosen as long as  $z$  remains non-negative.

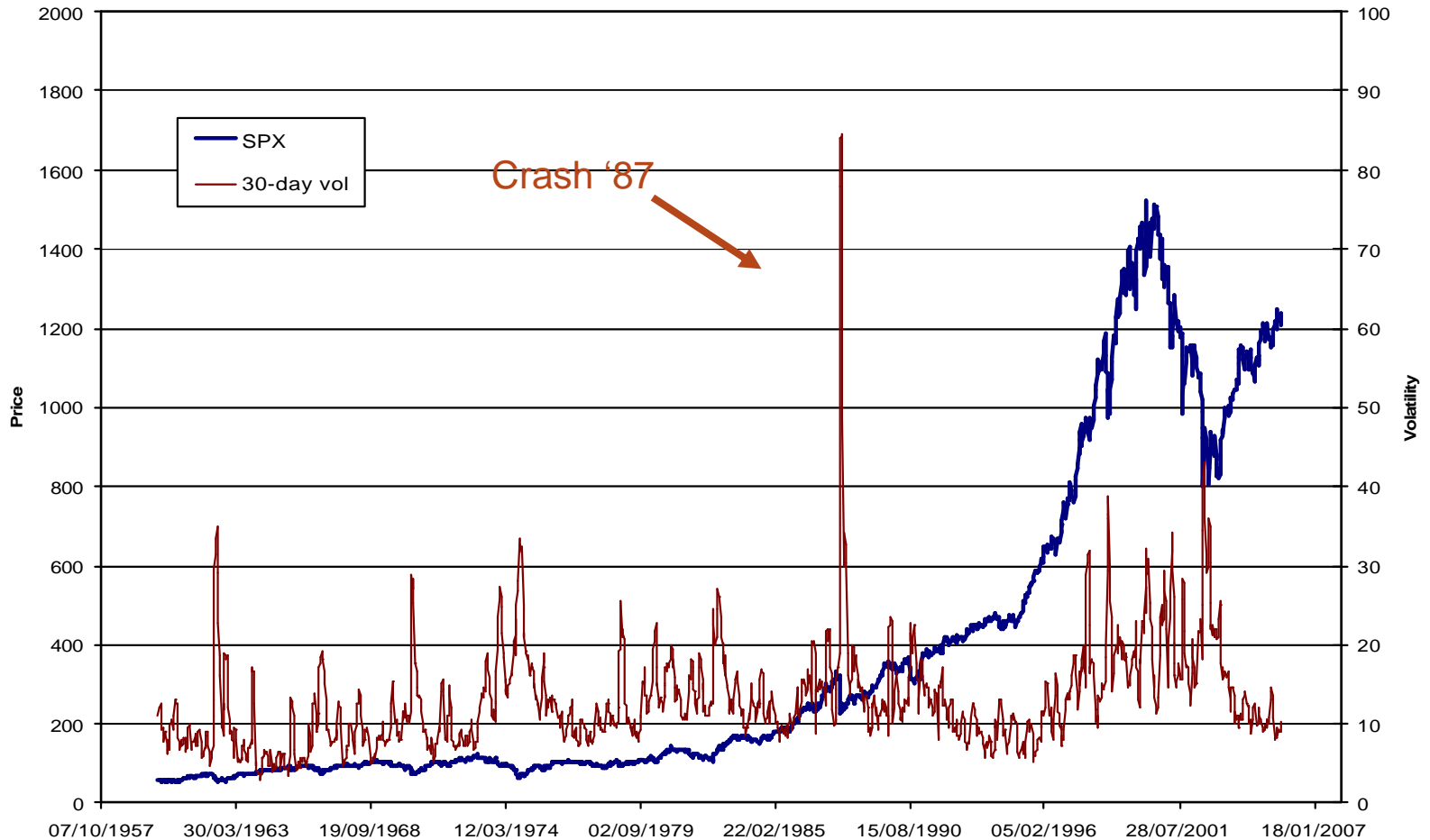
Mean-reversion level  $q$  is a positive martingale.



# Variance Curve Models

## Why mean-reversion?

SPX Spot level and 30-day realized volatility





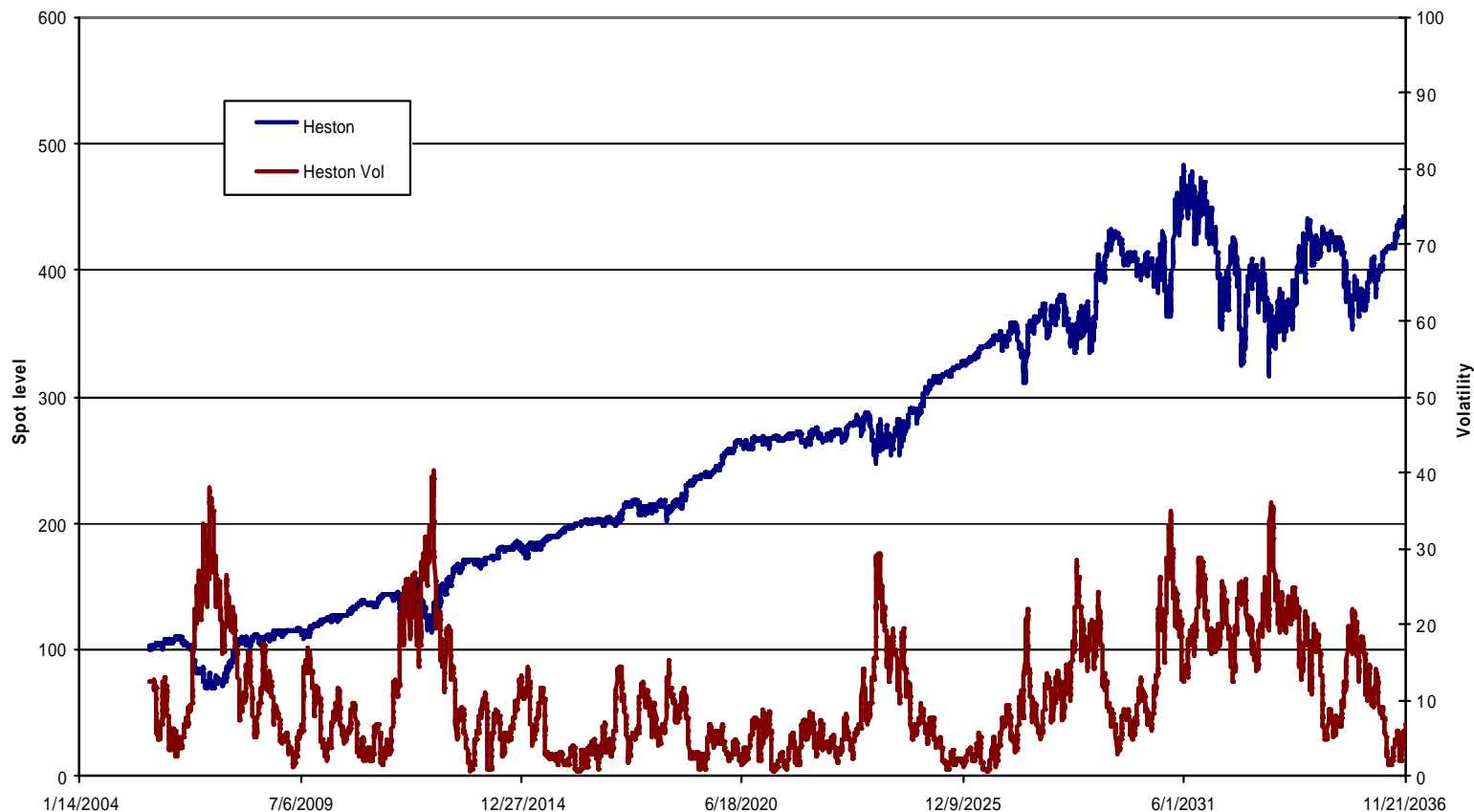
# Variance Curve Models

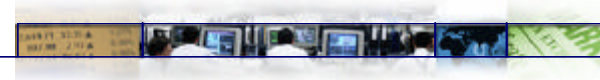
Why mean-reversion?

## Unconstrained Calibration

ShortVol	14.4%
LongVol	28.7%
RevSpeed	0.23
Correlation	-0.74
VolOfVol	26.3%

Heston path and 30-day realized volatility





# Variance Curve Models

## Variance Curve Functionals

### ■ Proposition

The observation that mean-reversion speeds must be constant holds for all polynomial-exponential functionals, i.e. if

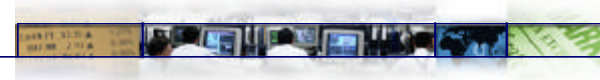
$$G(z_1, \dots, z_n, z_{n+1}, \dots, z_m; x) = \sum_{i=1}^n p_i(z; x) e^{-z_i x}$$

(where  $(p_i)_i$  are polynomials), then the first  $n$  components must be constant (cf. Filipovic 2001 for interest rates).

### ■ A similarly restrictive result can be shown for functionals of the form

$$G(z_1, \dots, z_n, z_{n+1}, \dots, z_m; x) = \exp \left\{ \sum_{i=1}^n p_i(z; x) e^{-z_i x} \right\}$$

- The parameters in the exponent come in pairs, where one is twice as large as the other (again Filipovic 2001).



# Variance Curve Models

## Variance Curve Functionals – Example linear mean-reversion

- Another example of the polynomial-exponential class is

$$G(z; x) = z_3 + (z_1 - z_2)e^{-kx} + (z_2 - z_3)\frac{k}{k-c}\left(e^{-cx} - e^{-kx}\right)$$

- A consistent factor model for this  $G$  must have the form

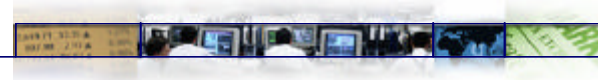
$$dZ_t^1 = \mathbf{k}(Z_t^2 - Z_t^1)dt + \mathbf{s}_1(Z_t)dW_t$$

$$dZ_t^2 = c(Z_t^3 - Z_t^2)dt + \mathbf{s}_2(Z_t)dW_t$$

$$dZ_t^3 = \mathbf{s}_3(Z_t)dW_t$$

which we call “double mean-reverting”.

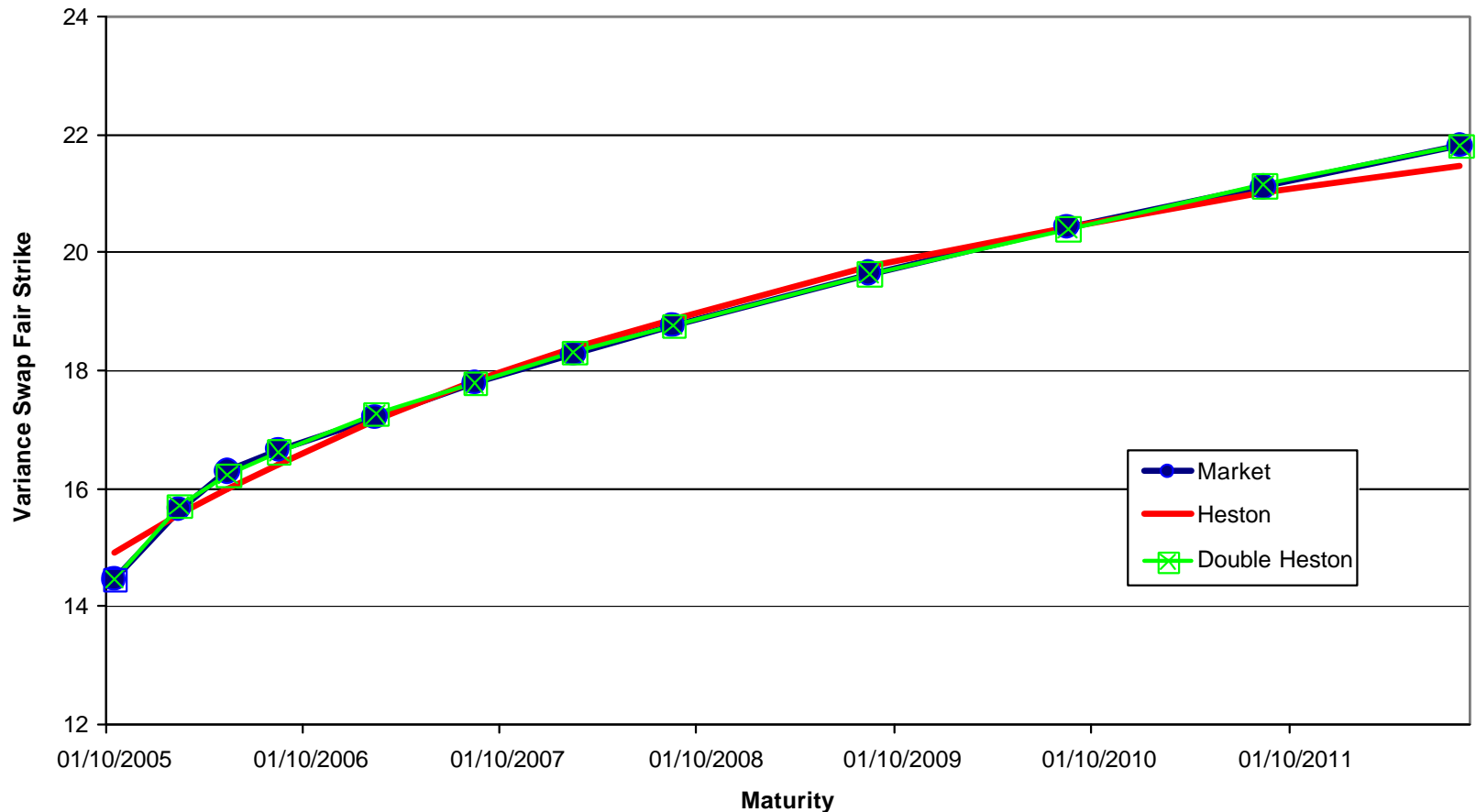
- Quite a good fit for most indices (at least during the course of the last year).



# Variance Curve Models

## Variance Curve Functionals – Example linear mean-reversion

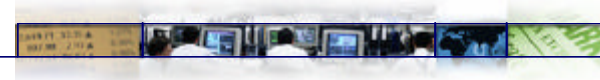
Variance Swap Term Structure .SPX 10/12/2005





# Hedging

Using Variance Curve Models to hedge Options on Variance



# Hedging

How to hedge with variance curve models

- We are back in our initial classical setting, i.e. we have decided to use a consistent variance curve model,

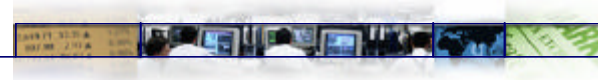
$$\hat{v}_t(x) = G(Z_t; x)$$

$$dZ_t = \mathbf{m}(Z_t)dt + \sum_{j=1, \dots, d} \mathbf{s}^j(Z_t) dW_t^j$$

$$\mathbf{z}_t := \hat{v}_t(0)$$

- We want to price and hedge an “option on realized variance” with (bounded) European payoff  $h$ . Its price process is given as

$$\mathbb{E} \left[ h \left( \int_0^T \mathbf{z}_s ds \right) \middle| F_t \right]$$



# Hedging

How to hedge with variance curve models

- Define the *variance swap price* function

$$\bar{G}(z; x) := \int_0^x G(z; y) dy$$

and assume that there exist constant  $0 < \varepsilon < t_1 < \dots < t_m$  such that

$$\bar{G}_{t_1, \dots, t_m}(z) := (\bar{G}(z; t_1), \dots, \bar{G}(z; t_m))$$

is invertible for all  $t_k := t_k - t$  for  $0 \leq \tau \leq \varepsilon$ .

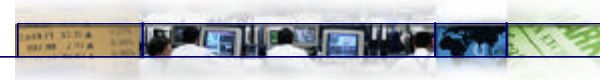
- This then allows to recover  $Z$  in any interval  $[a, b]$  by

$$Z_t = \bar{G}_{T_1-t, \dots, T_m-t}^{-1} \left( \underset{\substack{\uparrow \\ \text{Variance swap}}}{V_t(T_1)} - \underset{\substack{\uparrow \\ \text{Running realized variance}}}{V_t(t)}, \dots, V_t(T_m) - V_t(t) \right)$$

where  $T_k := a + t_k$ .

Variance swap

Running realized variance



# Hedging

How to hedge with variance curve models

- Due to the Markov-property of  $Z$ , we have

$$C_t(Z_t, V_t(t)) := E \left[ h \left( \int_0^T \mathbf{z}_s ds \right) \middle| Z_t; V_t(t) \right]$$

Running realized variance

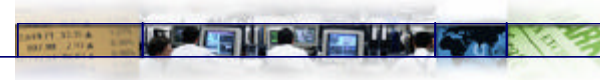


- To hedge this payoff in the interval  $[a, b]$ , we can write it due to our assumptions on  $G$  as

$$C_t(Z_t, V_t(t)) \equiv C_t(V_t(T_1), \dots, V_t(T_m), V_t(t))$$

such that (under the assumption that  $C$  is smooth enough)

$$dC_t(\dots) = \sum_{k=1}^m \partial_{V_k} C_t(\dots) dV_t(T_k)$$



# Hedging

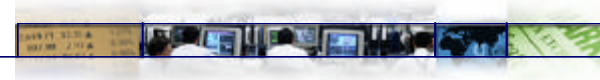
How to hedge with variance curve models

- This

$$dC_t(\dots) = \sum_{k=1}^m \partial_{V_k} C_t(\dots) dV_t(T_k)$$

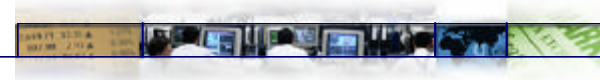
is the desired hedge of  $h$  in terms of variance swaps.

- For options on variance, this is a “natural” hedge.
  - It can also be used for standard options (a delta-term will appear).
  - For forward started options, correlation (skew) risk should be taken into account.
- In practise, the above “VarSwapDelta” hedging ratios are computed via bumping of the variance swap price.



# Variance Curves

Future



# Variance Curves

## Future

### ■ Challenges ahead

- Incorporation of stochastic interest rates and dividends (in particular long-term deals could exhibit strong exposure to stochastic interest rates).
- Jumps both in the underlying and the variance process (witness S&P return graph earlier).
- Correlation between the variance curves between different underlyings (our suspicion is that it is actually quite high).

- The dream, however, is to characterize the entire implied volatility surface or, equivalently, the “forward probability distribution” of  $S$ .



**Thank you very much for your attention.**

Working paper and at <http://www.math.tu-berlin.de/~buehler/>





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