Information-equivalence: On Filtrations Created by Independent Increments

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Summary. This article investigates filtrations created by the increments-processes of processes with independent increments: Suppose two processes create the same filtrations, when will the processes of their increments also create the same filtrations?

Our main result is that the processes of increments of two extremal continuous martingales with independent increments create the same filtrations if and only if either process admits a deterministic representation with respect to the other.

1 Introduction

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $X = (X_t)_{0 \leqslant t \leqslant T}$ a continuous process with horizon $T \in (0, \infty]$ and $\mathbb{F}(X) := (\mathcal{F}_t(X))_{t \leqslant T}$ with $\mathcal{F}_t(X) := \sigma(X_{s \leqslant t})$ the filtration created by X (we will assume that all σ-algebras are complete and that all filtrations are right-continuous). Note that we will use abbreviations similar to $X_{a \leqslant t \leqslant b} := (X_t)_{t \in [a,b]}$ or $\sigma(X_{t \geqslant a} - X_a) := \sigma(X_t - X_a;\, t \geqslant a)$ throughout the article (a and b are fixed numbers). For convenience, we will also assume that all processes in this article start at zero, i.e., $X_0 \equiv 0$.

We denote by $X^a = (X_t^a)_{t \leqslant T}$ the *increment-process of* X *at* a, i.e.,

$$X_t^a := X_{t \vee a} - X_a = X_t - X_{t \wedge a}.$$

Clearly, if X is a martingale, X^a is a martingale as well, adapted to both $\mathbb{F}(X)$ and $\mathbb{F}(X^a)$.

Now, given two processes X and $\widetilde{X}$ which create the same filtration, we may ask how the increments of these processes are related to each other.

For example, assume that two standard Brownian motions B and $\widetilde{B}$ create the same filtration. If we take $r \in (0, T)$, since both B^r and $\widetilde{B}^r$ are independent from $\mathcal{F}_r := \mathcal{F}_r(B) = \mathcal{F}_r(\widetilde{B})$, both $\sigma(B^r) \cap \mathcal{F}_r$ and $\sigma(\widetilde{B}^r) \cap \mathcal{F}_r$ are trivial.

But, under which circumstances are $\sigma(B^r)$ and $\sigma(\widetilde{B}^r)$ equal?

Consider the following definition:

Definition 1 (Information-similarity and -equivalence). *We call two processes* $X = (X_t)_{t \leqslant T}$ *and* $\widetilde{X} = (\widetilde{X}_t)_{t \leqslant T}$

1. ***information-similar***, *if* X^0 *and* $\widetilde{X}^0$ *create the same filtration, i.e.* $\mathbb{F}(X^0) = \mathbb{F}(\widetilde{X}^0)$, *and*
2. ***information-equivalent***, *if all their processes of increments are information similar, i.e.* $\mathbb{F}(X^a) = \mathbb{F}(\widetilde{X}^a)$ *holds for all* $a \in [0, T]$.

Additionally, for $I \subset [0, T]$, *we call* X *and* $\widetilde{X}$

3. ***information-semiequivalent (on*** I***)***, *if their processes of increments are information-similar for each* $a \in I \cup \{0\}$, *i.e.,* $\mathbb{F}(X^a) = \mathbb{F}(\widetilde{X}^a)$.

In the sequel, wherever suitable, we will abbreviate *information-similarity* by *i-similarity* and so forth. Moreover, we will sometimes write *ii* for "independent increments".

Let us now illustrate the preceding definition by the following example of two Brownian motions, which are i-similar but not i-equivalent:
Assume that B is a standard Brownian motion, choose an $r \in (0, T)$ and set

$$\widetilde{B}_t := \int_0^t h_s \, \mathrm{d}B_s \qquad \text{with} \qquad h_t := 1_{t \leqslant r} + 1_{t > r} \cdot \operatorname{sgn}^-(B_r) \tag{1}$$

where we define $\operatorname{sgn}^-(x) := 1_{x>0} - 1_{x \leqslant 0}$.

It can be shown easily that $\widetilde{B}$ is a Brownian motion and that B and $\widetilde{B}$ create the same filtration. However, their increments at r do not:

Assume that $\sigma(B^r) = \sigma(\widetilde{B}^r)$ and set $s^- := \operatorname{sgn}^-(B_r)$. By construction, we have $\widetilde{B}^r_t = s^- \cdot B^r_t$ for all $t > r$, i.e., we can identify s^- by using the information of two random variables measurable with respect to $\sigma(B^r)$. But s^- is non-trivial and independent of $\sigma(B^r)$. Since this is impossible, neither $\sigma(B^r) \subseteq \sigma(\widetilde{B}^r)$ nor $\sigma(B^r) \supseteq \sigma(\widetilde{B}^r)$.

Remark 1. In [2], Frank B. Knight calls two continuous processes X and $\widetilde{X}$ *past-and-future equivalent* if $\sigma(X_{s \leqslant t}) = \sigma(\widetilde{X}_{s \leqslant t})$ and $\sigma(X_{s \geqslant t}) = \sigma(\widetilde{X}_{s \geqslant t})$ for all $t \in [0, T]$.

Assume X is Markov and that $\widetilde{X}_t \in \sigma(X_{s \leqslant t}) \cap \sigma(X_{s \geqslant t})$. Then there are measurable functions g and f such that $\widetilde{X}_t = g_t(X_{s \leqslant t}) = \mathbb{E}[f_t(X_{s \geqslant t}) \mid X_{s \leqslant t}] = \mathbb{E}[f_t(X_{s \geqslant t}) \mid X_t] =: \widetilde{x}_t(X_t)$. Hence, two Markov-processes X and Y are *past-and-future equivalent*, if and only if we have a.s. $\widetilde{X}_t = \widetilde{x}_t(X_t)$ resp. $X_t = x_t(\widetilde{X}_t)$ for two suitable sequences of measurable functions x and $\widetilde{x}$ (compare lemma 1 in [2], [5], p. 113 and, for further related topics, section 17.3).

Clearly, past-and-future-equivalence and information-equivalence are different concepts; in this article, we study the latter and focus on processes with independent increments.

2 Equivalent Characterizations

Our first result shows that in case of processes with independent increments, information-equivalence is closely related to the properties of the time-reversed processes:

For a continuous ii-process $M = (M_t)_{t \leqslant T}$ with a finite time-horizon T, its *reverse process* $M'_t := M_T - M_{T-t}$ is also a continuous ii-process (note that the term *reverse process* is used differently in [2]). Then, given another ii-process $N = (N_t)_{t \leqslant T}$, we observe the following characterization:

Lemma 1. *Two continuous ii-processes M and N are i-equivalent if and only if both M and N and their time-reversed processes M' and N' are i-similar.*

Proof. We first note that M and N are i-equivalent iff for all fixed $a < b$ the relation

$$\sigma(M_b - M_{a \leqslant t \leqslant b}) = \sigma(N_b - N_{a \leqslant t \leqslant b}) \tag{2}$$

holds. Therefore, if M and N are i-equivalent, M' and N' are trivially i-similar.

Conversely, let us assume that M and N as well as their reverse processes are i-similar. We will show that (2) holds:

First, we choose a and b with $a < b \leqslant T$ and let $\mathcal{N} := \sigma(N_{a \leqslant t \leqslant b} - N_a)$. Then, since M and N are i-similar, we have

$$\mathcal{N} \subset \sigma(N_{a \leqslant t \leqslant b} - N_a;\, N_{u \leqslant a}) = \sigma(M_{a \leqslant t \leqslant b} - M_a;\, M_{u \leqslant a}).$$

On the other hand, because also M' and N' are i-similar,

$$\mathcal{N} \subset \sigma(N_{a \leqslant t \leqslant T} - N_a) = \sigma(M'_{t \leqslant T-a}) = \sigma(M_{b \leqslant u \leqslant T} - M_b;\, M_{a \leqslant t \leqslant b} - M_a).$$

Since $\sigma(M_{t \leqslant a})$, $\sigma(M_{a \leqslant t \leqslant b} - M_a)$ and $\sigma(M_{b \leqslant t \leqslant T} - M_b)$ are independent this yields that $\mathcal{N} \subset \sigma(M_{a \leqslant t \leqslant b} - M_a)$.

If we apply the same idea to $\sigma(M_{a \leqslant t \leqslant b} - M_a)$, we find that M and N are indeed i-equivalent by (2). □

Lemma 1 gives us a simple tool to check whether, say, two Brownian motions are i-equivalent. Indeed, remember our example of two Brownian motions defined in (1). There, for any $u \in (r, T)$, we find

$$\begin{aligned} \sigma(B'_{t \leqslant T-u}) &= \sigma(B_T - B_{u \leqslant t \leqslant T}) \\ &\neq \sigma\big(s^- \cdot (B_T - B_{u \leqslant t \leqslant T})\big) \\ &= \sigma\big(\widetilde{B}_T - \widetilde{B}_{u \leqslant t \leqslant T}\big) = \sigma\big(\widetilde{B}'_{t \leqslant T-u}\big), \end{aligned}$$

in correspondence with lemma 1.

Up to this point, M or N were general processes, not martingales. But now we want to link i-equivalence to the predictable representation property (PRP) of extremal martingales. What implication has information-equivalence for the representation of N by M?

Let $\mathbb{F}$ be a filtration, M a square-integrable martingale to $\mathbb{F}$ which starts at zero and define $L^2(M,\mathbb{F})$ as the space of $\mathbb{F}$-predictable $\mathbb{P}_M$-square-integrable integrands ($\mathbb{P}_M := \mathbb{P}\otimes \mathrm{d}\langle M\rangle$, compare [3], p. 137). Then, M is said to have the $\mathbb{F}$-PRP (or M is $\mathbb{F}$-extremal), iff for any other $\mathcal{L}^2(\mathbb{F})$-martingale N, there exists a unique integrand $H \in L^2(M,\mathbb{F})$ such that

$$N_t = N_0 + \int_0^t H_s \,\mathrm{d}M_s \tag{3}$$

(if we omit $\mathbb{F}$, we always refer to the filtration created by the process itself).

Remark 2. H is $\mathbb{P}_M$-a.e. not zero, if and only if N has the $\mathbb{F}$-PRP. In that case, we have $M_t = M_0 + \int_0^t H_s^{-1}\,\mathrm{d}N_s$ with $H_s^{-1} := (1/H_s)\cdot 1_{H_s\neq 0}$.

Proof. W.l.g. assume $T<\infty$. If H is $\mathbb{P}_M$-a.e. not zero and if $\mathrm{d}X = \xi\,\mathrm{d}M$, holds for a martingale X, then we have $\mathrm{d}X = \bar\xi\,\mathrm{d}N$ for $\bar\xi := (1/H)\cdot 1_{H\neq 0}\cdot\xi$.

Conversely, if N is $\mathbb{F}(M)$-extremal, consider the square-integrable $\mathbb{F}(M)$-martingale $X_t := \int_0^t 1_{H_s=0}\,\mathrm{d}M_s$. Then, there exists a unique K such that $X_t = \int_0^t K_s\,\mathrm{d}N_s = \int_0^t K_s H_s\,\mathrm{d}M_s$. Consequently, we find $\langle X,X\rangle = \int 1_{H_s=0}\cdot K_s H_s\,\mathrm{d}\langle M\rangle \equiv 0$, i.e., X_T is zero and we get $\mathbb{P}_M[H=0] = \mathbb{E}[X_T^2] = 0$. □

Remark 3. A continuous $\mathcal{L}^2$-martingale M with independent increments has a deterministic bracket. In particular, M is Gaussian.

Recall that the bracket of a continuous martingale with independent increments is deterministic, thus it is a time-changed Brownian motion and therefore Gaussian.

From now on, assume that M and N are two continuous centered $\mathbb{F}(M)$-extremal $\mathcal{L}^2$-martingales and that M has independent increments (we do not assert that N has independent increments nor that N creates the same filtration as M).

Then, our main theorem states:

Theorem 1. *M and N are information-equivalent if and only if H is a.e. deterministic, i.e. there is a version of H such that*

$$\mathbb{P}_M[H\neq H^*] = 0 \qquad \text{for} \qquad H_t^* := \mathbb{E}[H_t] \tag{4}$$

(we set H_t^* to zero if $\mathbb{E}[\,H_t\,]$ does not exist).

Theorem 1 shows that for such processes the term "information-equivalence" is quite reasonable: Being i-equivalent implies that we can construct one process from the other by applying a deterministic, i.e. foreseeable rule.

Note that this result once again yields that the Brownian motions B and B' as defined in (1) are not information-equivalent. An example for a Brownian motion, which is not even information-similar (but still extremal) is TANAKA's example, $\int_0^t \mathrm{sgn}^-(B_s)\,\mathrm{d}B_s$ (see [3], page 240).

Before we prove our theorem, we need some lemmata.

The first one is a trivial characterization of information-similarity, but nevertheless a convenient reference:

Lemma 2. *The processes M and N are information-similar if and only if H is also predictable with respect to the filtration $\mathbb{F}(N)$ created by N.*

Proof. If H is $\mathbb{F}(N)$-predictable, so is H^{-1}. Therefore $M_t = \int_0^t H_s^{-1}\,\mathrm{d}N_s$ has to be $\mathbb{F}(N)$-adapted. The reverse assertion is also trivial. $\square$

Since M is Gaussian, its increments are also Gaussian and therefore extremal on their own filtration (proposition 4.11 in [3], p. 213):

Lemma 3. *The process M^a is extremal on $\mathbb{F}(M^a)$ for all $a \in [0, T]$.*

For our second process N we find:

Lemma 4. *The process N^a is adapted to $\mathbb{F}(M^a)$, iff it is $\mathbb{F}(M^a)$-extremal.*

Proof. First, assume that N^a is adapted to $\mathbb{F}(M^a)$. Since N^a is an $\mathbb{F}(M^a)$-martingale, we find an $\mathbb{F}(M^a)$-predictable h such that $\mathrm{d}N^a = h\,\mathrm{d}M^a$. The $\mathbb{P}_M$-uniqueness of $H \in L^2(M, \mathbb{F}(M))$ given $\mathrm{d}N = H\,\mathrm{d}M$ yields that h is a version of $H|_{(a,T]}$. Because N is extremal, it is a.s. not zero and we can write $\mathrm{d}M^a = h^{-1}\,\mathrm{d}N^a$.

Reversely, if N^a has the $\mathbb{F}(M^a)$-PRP, we find $\mathrm{d}M^a = \bar{h}\,\mathrm{d}N^a$ for some $\bar{h} \in L^2(N^a, \mathbb{F}(M^a))$. The $\mathbb{P}_N$-uniqueness of $\mathrm{d}M = H^{-1}\,\mathrm{d}N$ yields with the usual arguments that $\mathrm{d}N^a = \bar{h}^{-1}\,\mathrm{d}M^a$. $\square$

Note that we found that if N^a is $\mathbb{F}(M^a)$-extremal, we have $\mathrm{d}N^a = \bar{h}^{-1}\,\mathrm{d}M^a$ for $\bar{h} \in L^2(N^a, \mathbb{F}(M^a))$. Hence, independence of M^a and $\mathcal{F}_a(M)$ shows that the increment N^a is independent of $\mathcal{F}_a(N)$.

Summing up the previous results yields the following technical lemma which we will use in the proof of theorem 1:

Lemma 5. *The processes M and N are information-semiequivalent on $I = \{t_1, \ldots, t_n\}$ for with $0 < t_1 < \cdots < t_n < T$ if and only if H has an $\mathbb{F}(N)$-predictable version which can be written as*

$$H_t = \sum_{k=0}^{n} H_t^k \cdot 1_{t \in (t_k, t_{k+1}]} \tag{5}$$

with $t_0 := 0$ and $t_{n+1} := T$ and where H^k is predictable w.r.t. $\mathbb{F}(M^{t_k})$.

Let us point out that this representation means that for $t \in (t_k, t_{k+1}]$, the value H_t depends solely on the "information" provided by the increment N^{t_k} i.e. M^{t_k} up to t. We could consider the points t_k as "resets" where the information carried by H until t_k is thrown away.

Proof. Assume first that M^{t_k} and N^{t_k} create the same filtrations for all $k = 0, \dots, n-1$. Then, H is $\mathbb{F}(N)$-predictable by lemma 2, and N^{t_k} is adapted to $\mathbb{F}(M^{t_k})$, on which M^{t_k} is extremal by lemma 3. The uniqueness of H w.r.t. to M therefore yields that H has a version such that $H|_{(t_k,T]} =: H^k \in L^2(M^{t_k}, \mathbb{F}(M^{t_k}))$.

Conversely, if H^{t_k} is predictable w.r.t. $\mathbb{F}(M^{t_k})$, N^{t_k} is adapted to $\mathbb{F}(M^{t_k})$ and therefore extremal by lemma 4 for all k. □

Given lemma 2 and 5, we can eventually turn to our proof of our main theorem 1. In principle, we would like to use lemma 5 and to take some kind of a limit from a finite number $t_1, \dots, t_n$ of points towards the entire interval $[0, T]$. In practise, this is what can be done via the properties of the stochastic integral since it can be defined as a limit of "simple" processes:

Proof of theorem 1. Trivially, if the deterministic process $H^* = (H^*_t)_{t \leqslant T}$ with $H^*_t := \mathbb{E}[H_t]$ is a version of H, once again lemma 2 yields that $\mathbb{F}(M^a)$ is equal to $\mathbb{F}(N^a)$ for every $a \in [0, T]$.

Now assume conversely, that M and N are i-equivalent.

Let $\mathcal{E}_0$ be the space of "simple integrands" vanishing at zero, i.e., of all processes $\xi_t = \sum_{k=-1}^n f_k \cdot 1_{t \in (t_k, t_{k+1}]}$ where each f_k is an $\mathcal{F}_{t_k}(M)$-measurable, bounded random variable and with a finite sequence $0 = t_0 < t_1 < \cdots < t_{n+1} = T$

Now, H^* is an $L^2(M)$-version of H iff for any $\xi \in \mathcal{E}_0$, the scalar product of ξ and H is equal to the product of ξ and H^*.

For this purpose, let us fix a $\xi_t = \sum_k f_k \cdot 1_{t \in (t_k, t_{k+1}]} \in \mathcal{E}_0$.

Then, because of the i-equivalence of M and N, M and N are i-semiequivalent on $I = \{t_1, \dots, t_n\}$, and lemma 5 asserts that there is a version of H which can be written as

$$H^\xi_t := \sum_{k=0}^n H^k_t \cdot 1_{t \in (t_k, t_{k+1}]},$$

where each H^k is $\mathbb{F}(M^{t_k})$-predictable and therefore in particular independent of $\mathcal{F}_{t_k}(M)$. This yields

$$\begin{aligned}
(H, \xi)_{L^2(M)} &= \left(H^\xi, \xi\right)_{L^2(M)} \\
&= \mathbb{E}\left[\sum_{k=0}^n \int_{t_k}^{t_{k+1}} H^k_t \cdot f_k \, \mathrm{d}\langle M \rangle_t\right] \\
&= \mathbb{E}\left[\sum_{k=0}^n f_k \cdot \mathbb{E}\left[\int_{t_k}^{t_{k+1}} H^k_t \, \mathrm{d}\langle M^{t_k} \rangle_t \;\middle|\; \mathcal{F}_{t_k}(M)\right]\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\sum_{k=0}^{n} f_k \cdot \mathbb{E}\left[\int_{t_k}^{t_{k+1}} H_t^k \,\mathrm{d}\langle M^{t_k}\rangle_t\right]\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{n} f_k \cdot \mathbb{E}_{\mathbb{P}_M}\left[H^k|_{[t_k,t_{k+1}]}\right]\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{n} f_k \cdot \mathbb{E}_{\mathbb{P}_M}\left[H|_{[t_k,t_{k+1}]}\right]\right] \\
&\overset{(*)}{=} \mathbb{E}\left[\sum_{k=0}^{n} f_k \cdot \int_{t_k}^{t_{k+1}} H_t^* \,\mathrm{d}\langle M\rangle_t\right] = \big(H^*, \xi\big)_{L^2(M)},
\end{aligned}
$$

where we have $(*)$ because M has a deterministic quadratic variation. □

At this stage, we want to stress the fact that i-equivalence of N and M already implies that N has independent increments: As shown in lemma 4, each N^a is extremal on $\mathbb{F}(M^a)$.

Lemma 6. *An extremal martingale X has independent increments if and only if each increment is extremal on its own filtration.*

Proof. If X^a is extremal, choose $A \in \mathcal{F}_T(X^a)$ and define the $\mathbb{F}(X^a)$-martingale $A_t^a := \mathbb{P}[A \,|\, X_{s\leqslant t}^a]$. We find a process $H^a \in L^2(X^a, \mathbb{F}(X^a))$ such that $dA^a = H^a \,\mathrm{d}X^a$. Since X is a martingale on $\mathbb{F}(Y)$, the extension $A_t := A_{t\vee a}^a$ is also a martingale on $\mathbb{F}(X)$, and we have $\mathbb{P}[A] = A_a^a = \mathbb{E}[A_T \,|\, \mathcal{F}_a(X)] = \mathbb{P}[A \,|\, \mathcal{F}_a(X)]$, ie independence of X^a and $\mathcal{F}_a(X)$.

Reversely, if Y^a is a $\mathbb{F}(X^a)$-martingale, apply the same extension as above and see, for $t > a$, that $\mathbb{E}[Y_t^a \,|\, \mathcal{F}_a(X)] = Y_a^a$, i.e., that Y is a $\mathbb{F}(X)$-martingale. Hence we can write $\mathrm{d}Y^a = H^a \,\mathrm{d}X^a$ and independence implies that $H^a \in L^2(X^a, \mathbb{F}(X^a))$. □

First of all, this lemma and its consequence that an i-equivalent N has obviously a deterministic quadratic variation, too, yields that $\mathrm{d}\langle N\rangle = H^2 \,\mathrm{d}\langle M\rangle$ is deterministic. Hence, the only source of randomness in H can only step from the sign of H.

A second observation is that the initial restriction to processes with independent increments does not seem too unnatural – at least regarding the property that their increments are extremal.

Note that only lemma 1 required that T is finite. In all other cases, we have proved all our claims also for an infinite time horizon. Hence, we can apply our result to a setting introduced by Tsirelson:

2.1 Noises

In [4], Boris Tsirelson used the description of "noise", to investigate the possibilities to linearize Brownian motions on Polish groups. We briefly repeat his notion, simplified for our setting, to add some comments which are results of the preceding sections.

Definition 2 (Noise). *A* noise *is defined as a family* $\mathbb{G} = (\mathcal{G}_t^s)_{s\leqslant t; s,t\in\mathbb{R}}$ *of* σ*-fields and a group* $\theta = (\theta_t)_t$ *of operators on the probability space such that*

1. θ_r *sends* $\mathcal{G}_t^s$ *onto* $\mathcal{G}_{t+r}^{s+r}$ *for all* $s \leqslant t$ *and* $r \in \mathbb{R}$,
2. $\mathcal{G}_s^r$ *and* $\mathcal{G}_t^s$ *are independent for all* $r \leqslant s \leqslant t$ *and*
3. $\mathcal{G}_s^r$ *and* $\mathcal{G}_t^s$ *generate* $\mathcal{G}_t^r$ *for all* $r \leqslant s \leqslant t$.

"Noise" as defined above is the property of a filtration. However, we want to link it to adapted and generating processes:

Definition 3 (Representation of a noise). *A* representation *of a noise is a family* $Z = (Z_t^s)_{s\leqslant t}$ *of random variables with values in a Polish group*[1] $(G, +)$ *verifying*

1. θ_r *sends* Z_t^s *to* Z_{t+r}^{s+r} *for* $s \leqslant t$ *and* $r \in \mathbb{R}$ *(i.e.,* $Z_t^s \circ \theta_r = Z_{t+r}^{s+r}$*),*
2. Z *is adapted to* $\mathcal{G}$*, i.e.* Z_t^s *is measurable w.r.t.* $\mathcal{G}_t^s$ *for* $s \leqslant t$,
3. $Z_s^r + Z_t^s = Z_t^r$ *for all* $r \leqslant s \leqslant t$ *and*
4. *for any* $\delta > 0$, $\mathbb{P}[|Z_t^s| \leqslant \delta] \to 1$ *for* $t \downarrow s$.[2]

We call such a representation continuous*, if*

5. *for any* $\delta > 0$, $\mathbb{P}[|Z_t^s| > \delta]/(t-s) \to 0$ *for* $t \downarrow s$

and we call it faithful *if*

6. $\mathcal{G}_t^0 = \sigma(Z_r^0; 0 \leqslant r \leqslant t)$ *for* $t \geqslant 0$.

The canonical example a continuous faithful representation of a noise is given by the standard Wiener space $\Omega = \mathbb{C}[\mathbb{R}]$, $\mathcal{F} = \mathbb{B}[\mathbb{R}]$ with the Wiener measure $\mathbb{P}$ and the shift-operator $\theta_t(\omega)(u) := \omega(t+u)$ for $\omega \in \Omega$.

Then, the coordinate process $X_t(\omega) := \omega(t)$ is a standard Brownian motion and obviously generates a noise with a continuous and faithful representation by virtue of our former convention $X_t^s := X_{t\vee s} - X_s$ and $\mathcal{G}_t^s := \mathcal{F}_t(X^s)$.

In fact, each $\mathbb{R}$-valued continuous faithful representation Z is in itself a sequence of continuous stationary increment processes with independent increments, i.e., a scaled Brownian motion with drift (eg. [5], page 115): Indeed, setting

$$Y_t := Z_t^0 1_{t\geqslant 0} - Z_0^t 1_{t<0}$$

we obtain a Brownian motion Y such that

[1] On a general Polish group (i.e., a topological group with a Polish metric, cf [4]) we define a Brownian motion as a continuous centered process with independent stationary increments; on $\mathbb{R}$, this coincides with the group of scaled Brownian motions.

[2] In a general Polish space (which has a metric d), we define $|x| := d(x, 0)$ where the symbol 0 denotes unity.

$$Y_t^s = Y_t - Y_s = \begin{cases} -Z_0^t + Z_0^s = Z_t^s & \text{for } s < t \leqslant 0, \\ Z_t^0 + Z_0^s = Z_t^s & \text{for } s \leqslant 0 < t, \\ Z_t^0 - Z_s^0 = Z_t^s & \text{for } 0 < s < t. \end{cases}$$

Note that this construction also shows how to construct noises on Ω.

Remark 4. Condition *6* in definition 3 is not symmetric since it does not yield a restriction on the way the fields $(\mathcal{G}_0^t)_{t<0}$ are generated. In order to obtain a similar "downward" condition, we shall consider the additional requirement

7. $\mathcal{G}_0^t = \sigma(Z_0^r;\, t \leqslant r \leqslant 0)$ for $t < 0$,

which rests on the former observation that $Z_0 := (Z_0^t)_{t;t\leqslant 0}$ is also a Brownian motion.

Given this extension, assume there is a continuous process $N = (N_t)_{t\in\mathbb{R}}$ such that $\mathcal{F}_t(N^0) = \mathcal{G}_t^0$ for $t \geqslant 0$ and $\sigma(N_0^r;\, t \leqslant r \leqslant 0) = \mathcal{G}_0^t$ for $t < 0$. Then,

Lemma 7. *The process N establishes a continuous faithful representation via $\tilde{Z}_t^s := N_t^s$ if and only if N^0 is information-equivalent to Z^0 and N_0 to Z_0.*

Proof. The requirements to provide a continuous faithful representation for $(\mathbb{G}, \theta)$ are obviously all met, except that N_t^s is supposed to be $\mathcal{G}_t^s$-measurable. This means that N is adapted to $\mathbb{G}$ in both directions (s resp. t).

For the case $s \geqslant 0$ fixed, this is by lemma 4 equivalent to N^0 and Z^0 being information-equivalent.

For the case $t < 0$ fixed, this is by the same reasoning equivalent to N_0 and Z_0 (as processes $(N_0^r)_{r;r\leqslant 0}$) being information-equivalent.

Since $\mathcal{G}_t^s = \mathcal{G}_0^s \vee \mathcal{G}_t^0$ for $s < 0 \leqslant t$, and $N_t^s = N_0^s + N_t^0$ this finishes the proof. □

Note that another possible modification of condition *6* in definition 3 is to request $\mathcal{G}_t^s = \sigma(Z_r^s;\, s \leqslant r \leqslant t) =: \mathcal{F}_t^s(Z)$ for all $s \leqslant t$. This gives rise to an extension of the term "information-equivalence" to processes with two time variables:

Definition 4. *Two processes $X = (X_t^s)_{s\leqslant t}$ and $Y = (Y_t^s)_{s\leqslant t}$ are called* information-equivalent *iff they create the same filtrations, i.e., $\sigma(X^s_{r;s\leqslant r\leqslant t}) = \sigma(Y^s_{r;s\leqslant r\leqslant t})$ for all $s \leqslant t$.*

Now, observe that $\mathcal{G}_t^s = \mathcal{F}_t^s(Z)$ for all $s \leqslant t$ is already implied when restricted to the cases where either $s = 0$ or $t = 0$ (as considered above), since $\mathcal{G}_t^s = \mathcal{G}_0^s \vee \mathcal{G}_t^0$ and $Z_t^s = Z_0^s + Z_t^0$ by definition and $\mathcal{F}_t^s(Z) = \sigma(Z_0^r + Z_u^0;\, s \leqslant r \leqslant 0 \leqslant u \leqslant t)$.

In other words, extending *6* by condition *7* already implies

Lemma 8. *Each continuous faithful representation N of $(\mathbb{G}, \theta)$ is information-equivalent to Z.*

Acknowledgement. Parts of this article are taken from the author's diploma thesis [1] (2001). I would like to thank Alexander Schied and Hans Föllmer, for their delightful help and support and I am also very grateful to Walter Schachermayer for fruitful conversations and to Laurent Nguyen-Ngoc for his continuing assistance.

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